Mixing of boundary Langevin dynamics for regulated four-dimensional Yang–Mills slabs

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Abstract

We study *regulated four-dimensional Yang–Mills theory on Euclidean slabs* $S_{t,L} := [0,t] \times \mathbb{T}_L^3$ at fixed thickness $t > 0$. A central theme is *kernel separation*: quantitative mixing of an auxiliary *boundary sampler* does not, by itself, imply mixing for the *Euclidean transfer kernel* $K_{t,\mathrm{Reg}}$ defined by disintegration of the slab endpoint law and governing Euclidean-time concatenation and transfer-operator statements.

Main theorem package (Wilson lattice slabs). For a finite-range Wilson lattice regulator family with compact gauge group G, we prove within this manuscript: (i) a fixed-window Doeblin minorisation for the projected transfer kernel, yielding *cylindrical* geometric contraction for $K_{t,\mathrm{Reg}}$ (Appendix F); (ii) in a KP (high-temperature) corridor $(0 < \beta \le \beta_\star(t,L))$ we prove a Wilson-intrinsic cross-slab maximal-correlation (polymer crossing) bound, hence L^2 mixing and a transfer spectral gap for $K_{t,\mathrm{Reg}}$ (Appendices G–H); and (iii) as a consequence, Euclidean-time exponential clustering and a time-axis transfer-operator gap for bounded gauge-invariant interior-supported cylinder slab observables (Definition 2.1). All constants in (i)–(iii) are uniform under UV refinement and auxiliary truncations at fixed (t, L). Moreover, within an L-uniform KP corridor $0 < \beta \le \beta_\star^\infty(t)$ we construct the spatial thermodynamic limit $L \to \infty$ for the Wilson slab family and show that the Euclidean-time clustering/gap conclusions persist for local (spatially supported on a fixed window) observables.

General template (Gaussian-reference regulators). Separately, we record an abstract implication scheme for slab specifications built over an abstract Wiener-space boundary reference, including quantitative mixing for a boundary Langevin sampler and a transport mechanism from boundary mixing to interior slab observables, under explicit stability/locality/moment inputs tracked in the constants ledger.

No continuum limit on $\mathbb{R}^4$ and no long-time limit $t \to \infty$ is constructed here; the only spatial infinite-volume result is the corridor thermodynamic limit $L \to \infty$ described above.

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1 Introduction

1.1 Motivation and scope

We emphasize that mixing of the auxiliary sampler kernel P_τ does *not* imply mixing of $K_{t,\mathrm{Reg}}$ without additional input; the two kernels play different roles.

Quantitative mixing and clustering bounds are central in constructive approaches to Euclidean quantum field theory [35, 34, 36], where one seeks to control correlation decay under limits in volume and ultraviolet regularisation. For gauge theories, a useful viewpoint is to work on a Euclidean slab $S_{t,L} = [0,t] \times \mathbb{T}_L^3$ and study the *boundary law* obtained by integrating out bulk degrees of freedom. At fixed slab thickness $t > 0$, this boundary law is a probability measure on a (typically infinite-dimensional) boundary configuration space.

Throughout, we fix the slab thickness $t > 0$. Most quantitative bounds in the paper are proved at fixed spatial size $L > 0$ and are uniform only under UV refinement and auxiliary truncations at fixed (t, L). In Section I we additionally construct the spatial thermodynamic limit $L \to \infty$ within an L-uniform KP corridor. No continuum limit on $\mathbb{R}^4$ and no long-time limit $t \to \infty$ is constructed here.

A standard route to decorrelation is to study a Markov dynamics that has the boundary law as an invariant measure and to prove explicit contraction estimates for its transition kernel. In the *Gaussian-reference template* part of this manuscript we introduce an auxiliary boundary Langevin sampler and develop quantitative contraction/mixing estimates for its kernel. In the *Wilson lattice* theorem package, however, the key object is instead the Euclidean transfer kernel $K_{t,\mathrm{Reg}}$ defined by endpoint disintegration; this is the kernel used for Euclidean-time concatenation and OS/transfer statements.

A second, logically distinct object also plays an essential role in OS-based constructions: the *Euclidean slab transfer kernel* $K_{t,\mathrm{Reg}}$, defined by disintegration of the joint law of the two boundary traces $(\mathrm{Tr}_-, \mathrm{Tr}_+)$ under the slab measure. This kernel governs concatenation of slabs in Euclidean time and is the correct input for time-axis exponential clustering and transfer-operator arguments.

Two clarifications address common misunderstandings:

- We work at *fixed* slab thickness $t > 0$. Uniformity in regulators is always at fixed t, while the behaviour as $t \to \infty$ is not addressed here.
- We distinguish *sampler time* (the Langevin parameter s) from *Euclidean slab time* (the thickness parameter t). Mixing of a sampler kernel P_s does *not* automatically imply mixing for the Euclidean transfer kernel $K_{t,\mathrm{Reg}}$. When we use Euclidean-time conclusions (time-axis exponential clustering, spectral gap), we do so under an explicit identification/comparison hypothesis.
- The paper is organized modularly: (boundary regularity) $\Rightarrow$ (drift + projected minorisation) $\Rightarrow$ (mixing for the boundary Langevin sampler), and separately (DLR/transfer structure) $\Rightarrow$ (time-axis exponential clustering and spectral consequences) once transfer-kernel mixing is available.

Remark 1.1 (Track firewall: Wilson lattice versus linear/Gaussian template). The *Wilson lattice track* in this paper is the compact-group boundary theory with boundary state space $\mathsf{B}_\partial := G^{E_\partial}$ (a compact manifold, not an additive vector space), Wilson local action, and finite-range locality. Any appearance of *Gaussian* objects (abstract Wiener spaces, Cameron–Martin norms, harmonic extension, Dirichlet-to-Neumann operators, or Fourier-mode linear contraction constants) is part of the *optional linear/template track* and is included only for motivation/bookkeeping. No Wilson proof (in particular, no proof of Theorem 2.14(ii)–(iii) and related Wilson-lattice corollaries) uses the linear/template machinery. Whenever differential notation is used in the Wilson track (e.g. "∇" or "Hessian"), it is understood intrinsically via left/right–invariant vector fields and the product bi-invariant Riemannian structure on G^{E_∂}.

1.2 Relationship to existing literature and what is new

The use of *transfer kernels/transfer matrices* and *reflection positivity* for lattice gauge theories is classical: already Osterwalder–Seiler established reflection positivity and the existence of a positive transfer matrix for lattice gauge theories, see e.g. [74, 76, 37], and Fröhlich's systematic treatment of reflection positivity for statistical-mechanical models [75].

Comparison to known results. Classical strong-coupling/cluster-expansion arguments on the lattice yield Euclidean-time correlation decay for *selected* families of observables (notably Wilson loops) at small inverse temperature β, and transfer-matrix/reflection-positivity methods provide a spectral interpretation of time translation. The theorem package proved here is stated instead at the *operator/kernel level*: it gives a quantitative contraction and spectral-gap mechanism for the *disintegration-defined* Euclidean transfer kernel $K_{t,\mathrm{Reg}}$ itself, with a constants ledger that is *uniform under UV refinement* at fixed (t, L) and, in an L-uniform corridor, survives the thermodynamic limit $L \to \infty$ (Section I). In particular, the paper isolates exactly which ingredients are needed to pass from strong-coupling expansions to transfer-kernel mixing and time-axis clustering, without conflating sampler time with Euclidean transfer time. A key methodological point emphasized here—sometimes implicit but rarely stated as an explicit dependency constraint—is the separation between: (a) *auxiliary sampling dynamics* designed to sample a boundary law (e.g. Langevin or heat-bath chains), and (b) the *Euclidean transfer kernel* $K_{t,\mathrm{Reg}}$ obtained by disintegration of the slab endpoint law, which is the kernel that enters slab concatenation, time-axis reflection positivity, and transfer-operator spectral statements. While transfer-matrix works use $K_{t,\mathrm{Reg}}$ (or its operator) as the primary object, stochastic-quantization and Markov-chain arguments in the physics literature sometimes blur the distinction between sampler time and Euclidean slab time. The present paper makes this separation explicit, packages it as a dependency map, and provides two *transfer-native* verification routes: a fixed-window Doeblin minorisation for the projected transfer kernel (Appendix F) and a functional-inequality route (mLSI/L^2 mixing) for $K_{t,\mathrm{Reg}}$ in a KP corridor (Appendices G–H). On the sampler side we provide a regulator-uniform weak-Harris argument with projected minorisation that avoids dimension blow-up (Appendix E), in the spirit of quantitative Harris theory [11, 9].

In the constructive-QFT literature, transfer-operator methods and OS reconstruction are standard tools [35, 34, 31, 32].

What is less explicit in much of the literature (and where misunderstandings often occur) is that *mixing of an auxiliary Markov sampler is not automatically mixing of the Euclidean transfer kernel*. Stochastic-quantisation approaches, for example, study a Langevin/SPDE dynamics whose invariant measure is (formally) the target Euclidean measure [55, 77], but Euclidean-time concatenation and transfer-operator statements are governed by the *endpoint-disintegration transfer kernel* $K_{t,\mathrm{Reg}}$. Our "kernel separation" emphasis makes this separation structural: transfer-kernel consequences are stated and proved only from transfer-side inputs.

Beyond this conceptual separation, the technical contribution of the present paper is a *quantitative, regulator-uniform bookkeeping* (constants ledger) together with two *in-paper* transfer-side verification mechanisms in a concrete Wilson lattice regime: a fixed-window Doeblin minorisation giving cylindrical contraction (Appendix F), and a KP corridor mechanism yielding a Wilson cross-slab maximal-correlation bound and hence L^2 mixing (Appendices G–H). These transfer-side mechanisms are then combined with a weak-Harris argument for the auxiliary boundary Langevin sampler (Appendix E) and with a transparent implication chain to time-axis clustering bounds for gauge-invariant observables at fixed regulator level.

1.3 Main contributions (informal)

The results can be summarised as follows.

C1. Boundary regularity estimates (fixed regulator) and a uniformity ledger. We derive C^2-regularity of the interacting boundary potential $U_{t,\mathrm{Reg}}$ along Cameron–Martin directions, with local (on bounded sets) Hessian bounds and a one-sided growth estimate (at fixed regulator), and we isolate the corresponding regulator-uniform requirements as explicit hypotheses for $\langle x, \nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}(x)\rangle_{\mathcal{H}}$.

C2. Quantitative mixing for a boundary Langevin sampler. Using the above bounds, we verify a Lyapunov drift condition and a projected minorisation condition for a fixed low-mode projection. This yields exponential contraction for a skeleton chain of the boundary Langevin sampler in a tailored Kantorovich distance $W_{1,\eta}^{(m)}$, at each fixed regulator; regulator-uniform consequences are obtained under explicit uniformity hypotheses on the constants (at fixed $t > 0$).

C3. Transfer-kernel consequences (conditional). The correct kernel for Euclidean-time concatenation is the slab transfer kernel $K_{t,\mathrm{Reg}}$. Under an explicit identification/comparison hypothesis that transfers the above contraction to $K_{t,\mathrm{Reg}}$ (or provides contraction for $K_{t,\mathrm{Reg}}$ directly), we obtain:

- time-axis exponential clustering for time-separated gauge-invariant local cylinder observables in an OS limit;
- a spectral-gap lower bound for the corresponding Euclidean-time transfer semigroup on the gauge-invariant sector.

1.4 Structure of the paper

Section 2 introduces the slab geometry, the regulated Yang–Mills model, the endpoint (transfer) structure, the boundary law, the boundary Langevin sampler, and states the main results precisely. Sections 3–6 establish the required regularity and growth bounds for $U_{t,\mathrm{Reg}}$. Section 7 verifies Lyapunov drift and projected minorisation and applies a weak Harris theorem to obtain exponential mixing for a skeleton chain of the sampler. Sections 9–13 formulate time-axis exponential clustering and spectral consequences under the transfer-kernel hypothesis. Technical auxiliary estimates are collected in appendices.

Notation snapshot. For quick reference, the following objects appear throughout (all at fixed slab thickness $t > 0$ and fixed spatial size $L > 0$):

Symbol	Meaning
$\mu^{\mathrm{bulk}}_{t,\mathrm{Reg}}$	regulated slab Gibbs measure on bulk fields
$\kappa_{t,\mathrm{Reg}}$	endpoint law $(\mathrm{Tr}_-, \mathrm{Tr}_+)_{\#}\mu^{\mathrm{bulk}}_{t,\mathrm{Reg}}$
$\nu_{t,\mathrm{Reg}} = \pi_{t,\mathrm{Reg}}$	boundary law at time 0 (first marginal of $\kappa_{t,\mathrm{Reg}}$)
$K_{t,\mathrm{Reg}}$	Euclidean transfer kernel (disintegration of $\kappa_{t,\mathrm{Reg}}$)
P_τ	boundary Langevin sampler kernel with invariant law $\pi_{t,\mathrm{Reg}}$
T_t	transfer operator $T_t f(b) = \int f(b')\, K_{t,\mathrm{Reg}}(b, \mathrm{d}b')$
$d_{m,\eta}$, $W^{(m)}_{1,\eta}$	bounded cost and associated Kantorovich distance

2 Setup and main results

2.1 Regulators, slab geometry, and observables

Fix a compact, connected Lie group G (in applications $G = \mathrm{SU}(N)$) with Lie algebra $\mathfrak{g}$. For $L > 0$ let $\mathbb{T}^3_L := (\mathbb{R}/L\mathbb{Z})^3$ and for $t > 0$ consider the Euclidean slab

$$S_{t,L} := [0,t] \times \mathbb{T}^3_L, \qquad \partial S_{t,L} = \{0\} \times \mathbb{T}^3_L \;\cup\; \{t\} \times \mathbb{T}^3_L.$$

We work with a family of *regulated* Yang–Mills measures on bulk gauge fields on $S_{t,L}$. The regulators include:

- a spatial volume regulator $L < \infty$;
- an ultraviolet regulator $\Lambda < \infty$ (e.g. Fourier cutoff, mollifier scale, or lattice spacing);
- optional auxiliary finite-dimensional regulators (e.g. mode truncations) used only for intermediate estimates.

We write $\mathrm{Reg} \equiv (L, \mathrm{Reg}_{\mathrm{UV}})$ for the full regulator tuple, where $\mathrm{Reg}_{\mathrm{UV}} \equiv (\Lambda, \dots)$ collects the ultraviolet cutoff and auxiliary truncations. Throughout we fix a slab thickness $t > 0$ and a spatial size $L > 0$. The adjective *regulator-uniform (at fixed (t, L))* means: constants are uniform in the UV cutoff and auxiliary truncations (and, when present, in intermediate finite-dimensional projections), with (t, L) treated as fixed parameters.

2.2 Concrete regulator family and coordinate realisation

The headline theorem in this paper is formulated for a concrete regulator family based on Wilson's lattice action on the slab $S_{t,L}$ with compact gauge group G and ultraviolet refinement (lattice spacing $a \downarrow 0$) at fixed (t, L). For context, we record separately an optional template-only Gaussian/harmonic-extension boundary framework; the Wilson-lattice proofs in this paper are intrinsic and do not rely on it. Accordingly, we work intrinsically on the compact group using left/right-invariant derivatives; local coordinate charts may be used only as a notational convenience, without imposing any hard cutoff

Group variables and reference measure. Let $\mathcal{E}_{t,L,a}$ denote the oriented edge set of a spacetime lattice discretisation of $S_{t,L}$ with spacing a. A configuration is a collection of link variables $U = (U_e)_{e \in \mathcal{E}_{t,L,a}} \in G^{\mathcal{E}_{t,L,a}}$ equipped with the product Haar measure $\mathrm{Haar}(\,\mathrm{d}U)$. We write $\mathsf{X}^{\mathrm{lat}}_{t,\mathrm{Reg}} := G^{\mathcal{E}_{t,L,a}}$ for the bulk link configuration space, and $\mathcal{B}_{t,\mathrm{Reg}} := G^{\mathcal{E}_\partial}$ for the boundary link configuration space at a time-slice. The Euclidean slab Gibbs weight is the Wilson plaquette action (with possible regulator-local counterterms)

$$\exp\{-S^{\mathrm{Wil}}_{t,L,a}(U)\}\,\mathrm{Haar}(\,\mathrm{d}U), \qquad S^{\mathrm{Wil}}_{t,L,a}(U) = \frac{1}{g^2}\sum_p (1 - \Re\,\mathrm{tr}\,U_p) + \cdots,$$

where U_p is the plaquette holonomy and "$\cdots$" denotes finite-range local terms allowed by the scheme.

Definition 2.1 (Wilson-lattice admissible class $\mathfrak{Obs}^{\mathrm{lat}}_{t,\delta}$)**.** Fix $\delta \in (0, t/2)$ and regulators $\mathrm{Reg} = (L, a, \Lambda, \dots)$. Let $\mathcal{V}_{t,L,a}$ be the vertex set and $\mathcal{E}_{t,L,a}$ the oriented edge set of the slab lattice. Write $s(e), t(e) \in \mathcal{V}_{t,L,a}$ for the source/target of an oriented edge e.

A *lattice gauge transformation* is $g = (g_v)_{v \in \mathcal{V}_{t,L,a}} \in G^{\mathcal{V}_{t,L,a}}$, acting on link configurations $U = (U_e)_{e \in \mathcal{E}_{t,L,a}} \in G^{\mathcal{E}_{t,L,a}}$ by

$$(U^g)_e \;:=\; g_{s(e)}\,U_e\,g^{-1}_{t(e)}.$$

A bounded observable $\mathcal{O} : G^{\mathcal{E}_{t,L,a}} \to \mathbb{R}$ is *gauge-invariant* if $\mathcal{O}(U^g) = \mathcal{O}(U)$ for all g.

Let $\mathcal{E}^{(\delta)}_{t,L,a} \subset \mathcal{E}_{t,L,a}$ be the set of edges whose geometric support lies in the interior time slab $[\delta, t-\delta] \times \mathbb{T}^3_L$. A bounded observable $\mathcal{O}$ is *δ-interior-supported* if it is cylindrical and depends only on finitely many edge variables $\{U_e : \ e \in W\}$ for some finite set $W \subset \mathcal{E}^{(\delta)}_{t,L,a}$.

We write $\mathfrak{Obs}^{\mathrm{lat}}_{t,\delta}$ for the class of bounded, gauge-invariant, δ-interior-supported cylinder observables.

Lattice Wilson loops. Let $\gamma = (e_1^{\sigma_1}, \dots, e_n^{\sigma_n})$ be an oriented closed edge path contained in $[\delta, t-\delta] \times \mathbb{T}^3_L$, with $\sigma_j \in \{\pm 1\}$. Define the holonomy

$$U_\gamma(U) \;:=\; \prod_{j=1}^{n} U_{e_j}^{\sigma_j}, \qquad U_e^{-1} \text{ used when } \sigma_j = -1.$$

For a unitary representation ρ of G, the associated Wilson loop observable is

$$W_{\rho,\gamma}(U) \;:=\; \frac{1}{\dim \rho}\,\Re\,\mathrm{tr}\big(\rho(U_\gamma(U))\big),$$

which lies in $\mathfrak{Obs}^{\mathrm{lat}}_{t,\delta}$.

No auxiliary small-field cutoff is needed in the Wilson corridor. For the Wilson lattice gauge model with compact gauge group G, the configuration space G^E is compact and the Wilson plaquette potential is a smooth class function of the plaquette holonomy. All derivative bounds needed in the KP/cluster-expansion corridor can therefore be taken *globally* using left/right-invariant directional derivatives on G (equivalently, derivatives along the Lie algebra via left translation), with constants depending only on G and the chosen finite-range local potential, not on any chart radius. Accordingly, the main Wilson/KP results below do *not* impose any auxiliary "small-field"/chart cutoff.

Optional Euclidean-reference template. For certain *linear* or gauge-fixed discretisations (not the Wilson compact-group model), one can represent the regulated slab law using a Gaussian bridge reference measure on a linear boundary space and write the interacting law as a tilt by a local potential. We include this Euclidean/Gaussian calculus as an optional bookkeeping template; none of the concrete Wilson slab theorems below use it.

2.3 Model dictionary and notational conventions

The manuscript is organised around a *single concrete theorem package* for Wilson lattice Yang–Mills slabs with compact gauge group G, and an *optional abstract template* that is included only for bookkeeping of general Markov/OS mechanisms.

- **Concrete Wilson setting (used in the main theorems).** Boundary data are boundary link configurations $b \in \mathsf{B}_\partial := G^{E_\partial}$ with reference measure Haar. All derivatives in the Wilson corridor are interpreted intrinsically using left-invariant vector fields and the product bi-invariant Riemannian structure on G^{E_∂}; no auxiliary chart or "small-field" cutoff is assumed.
- **Optional Euclidean/Gaussian template (not used in the Wilson proofs).** Some sections develop a linear/Gaussian reference calculus on an abstract boundary space $(E, \mathcal{H}, \mu_0)$. Whenever this template is invoked we write the boundary variable as $x \in E$ (unrelated to the Wilson boundary field $b \in \mathsf{B}_\partial = G^{E_\partial}$; in Section 4 endpoint traces are denoted $x_\pm$).
- **Endpoint partition functions and effective potentials (both settings).** For fixed boundary datum $b \in G^{E_\partial}$ (Wilson) or $x \in E$ (template), let $Z_{t,\mathrm{Reg}}(\,\cdot\,)$ denote the one-slab conditional partition function obtained by integrating out interior/bulk degrees of freedom with the endpoint(s) fixed. We write $U_{t,\mathrm{Reg}} = -\log Z_{t,\mathrm{Reg}} + \mathrm{const}$ for the centred one-boundary effective potential, and $U^{(2)}_{t,\mathrm{Reg}}(b,b') = -\log Z^{(2)}_{t,\mathrm{Reg}}(b,b') + \mathrm{const}$ for the centred two-endpoint potential.
- **Transfer kernel.** The Euclidean transfer kernel $K_{t,\mathrm{Reg}}$ is always defined by endpoint disintegration of the two-endpoint law $\kappa_{t,\mathrm{Reg}}$ (Definition 2.7); it is a Markov kernel on the relevant boundary state space (in particular, on G^{E_∂} for Wilson slabs).

- **What is expanded in Appendix H.** The KP corridor argument is *not* an assumption that an already-integrated boundary potential is finite-range in the boundary variables. Instead, Appendix H applies a polymer/cluster expansion to the *local slab interaction functional* and derives the required bounds for $\log Z$ (and hence for U and mixed derivatives of $U^{(2)}$) by termwise differentiation of the convergent expansion.

2.4 Optional template: Boundary configuration space and Gaussian reference structure

The boundary law will be a probability measure on a (typically infinite-dimensional) boundary configuration space. We model it as an *abstract Wiener space* $(E, \mathcal{H}, \mu_{0,t,\mathrm{Reg}})$:

- E is a separable Banach space;
- $\mathcal{H} \subset E$ is a densely and continuously embedded separable Hilbert space (the Cameron–Martin space);
- $\mu_{0,t,\mathrm{Reg}}$ is a centered Gaussian measure on E with Cameron–Martin space $\mathcal{H}$.

The concrete realisation depends on the chosen gauge-fixing and ultraviolet regularisation. For the purposes of this paper, we only use the abstract Wiener structure plus the existence of a family of bounded, self-adjoint covariance operators $(C^0_{t,\mathrm{Reg}})$ associated with $\mu_{0,t,\mathrm{Reg}}$ (equivalently, the quadratic form of $(C^0_{t,\mathrm{Reg}})^{-1}$ on $\mathcal{H}$). We write the Cameron–Martin norm by $\|\cdot\|_{\mathcal{H}}$ and the dual pairing by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

Remark 2.2 (On gauge-fixing and boundary variables). The boundary variable $x \in E$ should be read as the (regulated) boundary trace of the bulk field in a fixed gauge. All results are formulated for gauge-invariant observables and are insensitive to the particular gauge-fixing, provided the resulting endpoint laws admit the structural properties stated below.

2.5 Endpoint law and Euclidean transfer kernel

Fix $t > 0$ and regulators $\mathrm{Reg} = (L, \Lambda, \ldots)$. Let $\mu^{\mathrm{bulk}}_{t,\mathrm{Reg}}$ denote a regulated, local slab Gibbs measure on bulk fields over $S_{t,L}$ (Yang–Mills) with a local stable action and the reflection/Markov properties used below (i.e. a standard regulated Yang–Mills slab measure, such as Wilson lattice gauge theory, formulated in a fixed local chart; in this paper we do not use BRST/ghost structure). Let $\mathcal{B}_{t,\mathrm{Reg}}$ denote the boundary state space at Euclidean times 0 and t. In the Wilson lattice track, $\mathcal{B}_{t,\mathrm{Reg}} = G^{\mathcal{E}_\partial}$ is the space of boundary link variables (on the oriented spatial edge set $\mathcal{E}_\partial$ of a time-slice of the slab lattice). In the optional template track, $\mathcal{B}_{t,\mathrm{Reg}} = E$. Let Tr_- and Tr_+ denote the boundary restriction maps at time 0 and time t, respectively, defined on the underlying bulk configuration space of $\mu^{\mathrm{bulk}}_{t,\mathrm{Reg}}$ and taking values in $\mathcal{B}_{t,\mathrm{Reg}}$.

Definition 2.3 (Endpoint law and boundary marginal)**.** The (regulated) *endpoint law* is the pushforward probability measure on $\mathcal{B}_{t,\mathrm{Reg}} \times \mathcal{B}_{t,\mathrm{Reg}}$,

$$\kappa_{t,\mathrm{Reg}} \;:=\; (\mathrm{Tr}_-, \mathrm{Tr}_+)_{\#}\mu^{\mathrm{bulk}}_{t,\mathrm{Reg}}.$$

Its first marginal is the (regulated) *boundary law at time* 0:

$$\nu_{t,\mathrm{Reg}} \;:=\; (\mathrm{Tr}_-)_{\#}\mu^{\mathrm{bulk}}_{t,\mathrm{Reg}}.$$

Definition 2.4 (Normalised trace law)**.** The (regulated) *normalised trace law* at time 0 is the first marginal of the endpoint law:

$$\pi_{t,\mathrm{Reg}} \;\coloneqq\; \nu_{t,\mathrm{Reg}}.$$

When the regulator is understood we write simply π.

Remark 2.5 (Notation ν versus π)*.* We write $\nu_{t,\mathrm{Reg}}$ for the boundary law when emphasising its origin as a marginal of the slab measure, and $\pi_{t,\mathrm{Reg}}$ when emphasising its role as the invariant/reversible measure for a Markov kernel (sampler or transfer). In this paper these coincide by definition.

Remark 2.6 (Boundary state space and two model tracks)*.* Statements about the Euclidean transfer kernel and its time-axis consequences are formulated on an abstract boundary state space $\mathcal{B}_{t,\mathrm{Reg}}$ associated with the regulator tuple Reg. In the concrete Wilson lattice family, $\mathcal{B}_{t,\mathrm{Reg}}$ is the compact manifold of boundary link configurations $G^{\mathcal{E}_\partial}$ equipped with product Haar reference measure. In the Gaussian template track, $\mathcal{B}_{t,\mathrm{Reg}} = E$ is the underlying Banach space of an abstract Wiener space $(E, \mathsf{H}_0, \mu_{0,t,\mathrm{Reg}})$. Template-only assumptions that rely on the Gaussian reference are explicitly marked as such and are not used in the Wilson theorem package.

Definition 2.7 (Euclidean transfer kernel)**.** A *Euclidean transfer kernel* is any Markov kernel $K_{t,\mathrm{Reg}}(x_0,\,\mathrm{d}x_1)$ on $\mathcal{B}_{t,\mathrm{Reg}}$ which is a disintegration of $\kappa_{t,\mathrm{Reg}}$ with respect to its first marginal $\nu_{t,\mathrm{Reg}}$, i.e.

$$\kappa_{t,\mathrm{Reg}}(\,\mathrm{d}x_0,\,\mathrm{d}x_1) \;=\; \nu_{t,\mathrm{Reg}}(\,\mathrm{d}x_0)\,K_{t,\mathrm{Reg}}(x_0,\,\mathrm{d}x_1).$$

In particular, $K_{t,\mathrm{Reg}}$ is the one-step kernel governing concatenation of slabs in Euclidean time.

Uniqueness. If $K_{t,\mathrm{Reg}}$ and $K'_{t,\mathrm{Reg}}$ are two disintegrations of $\kappa_{t,\mathrm{Reg}}$ with respect to $\nu_{t,\mathrm{Reg}}$, then $K_{t,\mathrm{Reg}}(x,\cdot) = K'_{t,\mathrm{Reg}}(x,\cdot)$ for $\nu_{t,\mathrm{Reg}}$-a.e. x. All statements below depend only on this $\nu_{t,\mathrm{Reg}}$-a.e. equivalence class.

Remark 2.8 (Why this kernel matters)*.* time-axis exponential clustering and transfer-operator spectral statements are naturally formulated in terms of the Markov chain on boundary data driven by $K_{t,\mathrm{Reg}}$ (or its OS-limit analogue). A sampler dynamics (such as the Langevin process defined below) is conceptually useful, but its mixing properties do not automatically transfer to $K_{t,\mathrm{Reg}}$ unless one proves an identification or comparison principle.

2.6 Template track: boundary law as a Gibbs tilt of a Gaussian reference

Template-only. The assumptions in this subsection are used only in the Gaussian reference template track. They are not invoked in the Wilson lattice theorem package, where the natural reference measure is Haar on $G^{\mathcal{E}_\partial}$.

We assume that the boundary law $\nu_{t,\mathrm{Reg}}$ is absolutely continuous with respect to the Gaussian reference measure $\mu_{0,t,\mathrm{Reg}}$ on $(E, \mathcal{H})$.

Assumption 2.9 (Template track: Gaussian reference and boundary density)**.** For each fixed $t > 0$ and regulator tuple Reg, the boundary law $\nu_{t,\mathrm{Reg}}$ satisfies $\nu_{t,\mathrm{Reg}} \ll \mu_{0,t,\mathrm{Reg}}$, and there exists a measurable function $U_{t,\mathrm{Reg}} : E \to \mathbb{R} \cup \{+\infty\}$ such that

$$\nu_{t,\mathrm{Reg}}(\,\mathrm{d}x) \;=\; \frac{1}{\mathcal{Z}_{t,\mathrm{Reg}}}\exp\bigl(-U_{t,\mathrm{Reg}}(x)\bigr)\,\mu_{0,t,\mathrm{Reg}}(\,\mathrm{d}x), \qquad \mathcal{Z}_{t,\mathrm{Reg}} = \int \exp(-U_{t,\mathrm{Reg}})\,\mathrm{d}\mu_{0,t,\mathrm{Reg}}. \tag{1}$$

The potential $U_{t,\mathrm{Reg}}$ is defined up to an additive constant, which we fix by $U_{t,\mathrm{Reg}}(0) = 0$ whenever finite.

Remark 2.10 (Relation to bulk conditional partition functions). In concrete regulated constructions, $U_{t,\mathrm{Reg}}$ is often obtained as a (renormalised) log-partition function by integrating out bulk fields conditioned on boundary data. In this paper we work with the abstract representation (1) and verify structural properties of $U_{t,\mathrm{Reg}}$ using differentiation identities in Section 3 and model-specific estimates later on.

2.7 Boundary Langevin dynamics (sampler)

Let

$$\Phi_{t,\mathrm{Reg}}(x) \;:=\; \frac{1}{2}\,\Big\langle x, (C^0_{t,\mathrm{Reg}})^{-1}x\Big\rangle_{\mathcal{H}} + U_{t,\mathrm{Reg}}(x), \tag{2}$$

Remark 2.11 (Whitened coordinates for the Gaussian boundary law). The quadratic part in (2) is written with the free boundary covariance $C^0_{t,\mathrm{Reg}}$. Equivalently, one may work in *whitened* Cameron–Martin coordinates $\tilde{x} = (C^0_{t,\mathrm{Reg}})^{-1/2}x$, in which the free Gaussian law becomes standard and the Ornstein–Uhlenbeck part of the drift is simply $-\tilde{x}$. All statements are invariant under this linear isometry; in particular, whenever we write the sampler in the form $\mathrm{d}B_s = -(B_s + \nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}(B_s))\,\mathrm{d}s + \sqrt{2}\,\mathrm{d}W_s$ (Appendix E), this is understood in whitened coordinates. In unwhitened coordinates, replace B_s by $(C^0_{t,\mathrm{Reg}})^{-1}B_s$ in the OU drift.

interpreted as a function on E which is differentiable along Cameron–Martin directions.

Definition 2.12 (Smoothness along Cameron–Martin directions)**.** Let (E, H, μ) be an abstract Wiener space. A function $F : E \to \mathbb{R}$ is said to be C^k *along Cameron–Martin directions* if for every $x \in E$ the map $\mathsf{H} \ni h \mapsto F(x+h)$ is k-times Fréchet differentiable on H, and the derivatives $D^j_{\mathsf{H}}F(x)$ extend to bounded j-linear forms on H that depend measurably on x.

The *boundary Langevin dynamics* is the E-valued Markov process $(B_s)_{s\geq 0}$ solving the (formal) SDE

$$\mathrm{d}B_s = -\nabla_{\mathcal{H}}\Phi_{t,\mathrm{Reg}}(B_s)\;\mathrm{d}s + \sqrt{2}\;\mathrm{d}W_s, \tag{3}$$

where W_s is a cylindrical Brownian motion in $\mathcal{H}$ realised on E (as standard for abstract Wiener spaces), and $\nabla_{\mathcal{H}}$ denotes the gradient along Cameron–Martin directions.

Remark 2.13 (Reversibility/invariance). At the level of fixed finite-dimensional regulators, (3) is the usual overdamped Langevin dynamics with invariant density proportional to $\exp(-\Phi_{t,\mathrm{Reg}})$. In the abstract Wiener setting, the process is understood in the standard weak/mild sense and is designed to be reversible with respect to the boundary law $\nu_{t,\mathrm{Reg}}$ in (1).

2.8 Low-mode projection and the distance used for contraction

Because the dynamics (3) generally smooths only partially in infinite dimensions, we use a fixed finite-dimensional *low-mode* projection to formulate a minorisation condition. Let $\Pi_m : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection onto an m-dimensional subspace $\mathcal{H}_m \subset \mathcal{H}$. We extend Π_m to a measurable map on E (still denoted Π_m) using the canonical embedding $\mathcal{H} \hookrightarrow E$.

Fix parameters $\eta \in (0, 1]$ and $R > 0$. Define a bounded, lower semicontinuous cost $d_{m,\eta}$: $E \times E \to [0, 1]$ by

$$d_{m,\eta}(x, y) \;:=\; \min\Big\{1,\; \|(I - \Pi_m)(x - y)\|_E + \eta\; \|\Pi_m(x - y)\|_{\mathcal{H}}\Big\}, \tag{4}$$

where $\|\cdot\|_E$ is the Banach norm on E. (Any equivalent choice of bounded cost controlling both high-mode displacement in E and low-mode displacement in $\mathcal{H}$ is acceptable; we fix (4) for definiteness.)

Wilson-track guardrail. The difference $x-y$ in (4) is specific to the optional linear/template track. In the Wilson lattice track the boundary space is $\mathsf{B}_\partial = G^{E_\partial}$ and distances are formulated intrinsically using bounded metrics built from the bi-invariant Riemannian (or geodesic) distance on G and products thereof.

For probability measures μ, ν on E, define the associated Kantorovich distance

$$W_{1,\eta}^{(m)}(\mu,\nu) \;:=\; \inf_{\pi\in\Pi(\mu,\nu)} \int_{E\times E} d_{m,\eta}(x,y)\,\pi(\,\mathrm{d}x,\,\mathrm{d}y), \tag{5}$$

where $\Pi(\mu,\nu)$ denotes the set of couplings. We will show that for a suitable skeleton time $s_\star > 0$, the Markov kernel $P_{s_\star}$ of (3) contracts in $W_{1,\eta}^{(m)}$ at an exponential rate, with constants uniform in Reg (at fixed $t > 0$).

2.9 Two theorem packages: a concrete lattice regime and an abstract framework

The paper is organised around two complementary theorem packages.

We now state the headline result of the paper: a *concrete* regulator family in which both (i) the boundary Langevin sampler mixes quantitatively and (ii) the Euclidean slab transfer kernel $K_{t,\mathrm{Reg}}$ admits *in-paper* mixing mechanisms (Doeblin on a fixed window, and a KP/high-temperature corridor implying mLSI/L^2 mixing). This is the regime in which the "constants ledger" becomes a theorem rather than a wish list.

For completeness, we also record later a *general template* for arbitrary local slab Gibbs specifications: under explicit standing inputs (Gaussian reference, stability/non-degeneracy, finite-range locality, and moment/response bounds), one obtains regulator-uniform structural estimates for the boundary potential and regulator-uniform weak-Harris mixing for the boundary sampler. Those abstract results are intended as a reusable module *beyond* the concrete lattice corridor treated here.

Theorem 2.14 (Main theorem package: Wilson lattice Yang–Mills slabs (fixed t; UV refinement; $L \to \infty$ in the corridor))**.** *Fix $t > 0$ and $L > 0$ and a compact connected Lie group G. Items* (i)–(iii) *below are at fixed L, while item* (iv) *addresses the thermodynamic limit $L \to \infty$ within an L-uniform KP corridor. Consider Wilson lattice Yang–Mills on the slab $S_{t,L}$ with lattice spacing a (UV refinement $a \downarrow 0$ at fixed (t, L)), compact gauge group G, and Wilson plaquette action (allowing finitely many local counterterms of the same finite range). No auxiliary small-field/chart cutoff is imposed in the Wilson/KP corridor; compactness of G provides global smoothness and uniform derivative bounds.*

Assume that the locality/regularity input of Assumption H.2 holds for this Wilson regulator family (verified in Lemma H.4 below), and fix a coupling $0 < \beta \le \beta_\star(t, L)$ in the KP (high-temperature) corridor of Theorem G.1 (proved in Appendix G using the KP machinery of Appendix H). In particular, Theorem G.1(b)–(c) *yields a Wilson cross-slab contraction constant $q_W(t, L) \in (0, 1)$ such that $\|K_{t,\mathrm{Reg}}\|_{L^2_0(\pi_{t,\mathrm{Reg}})\to L^2_0(\pi_{t,\mathrm{Reg}})} \le q_W(t, L)$ uniformly in UV refinement at fixed (t, L).*

Then, for this Wilson lattice family:

(i) Cylindrical transfer-kernel contraction on a fixed window (what Doeblin actually buys). *Let W be the fixed finite window of Appendix F and let Π_W be the corresponding coordinate projection. In this lattice family, Assumption 2.18 holds by choosing the finite-rank projection Π_m there to be Π_W (and the corresponding bounded cost d_η on G^W); see Corollary 2.29. Equivalently, for every bounded* cylindrical *observable f depending only on the window coordinates $\Pi_W b$, and all $x, y \in \mathcal{B}_{t,\mathrm{Reg}}$,*

$$\left|K_{t,\mathrm{Reg}}^n f(x) - K_{t,\mathrm{Reg}}^n f(y)\right| \;\le\; 2\,\|f\|_\infty\,(1-\varepsilon_W)^n, \qquad n \in \mathbb{N},$$

where ε_W is the Doeblin constant from Lemma F.1 (uniform in UV refinement at fixed (t, L) and fixed local-potential parameters). All later uses of the Doeblin route in this paper are of this cylindrical/local form (in particular for observables supported away from time boundaries).

(ii) Transfer-kernel L^2 mixing in the corridor. *In the KP corridor, Corollary 2.25 yields the Wilson-intrinsic $L^2(\pi_{t,\mathrm{Reg}})$ mixing estimate (212) and transfer spectral-gap bound (213) for the disintegration-defined transfer kernel $K_{t,\mathrm{Reg}}$, with constants uniform in the UV refinement at fixed (t, L). Here $q_W(t, L) < 1$ is the crossing-polymer/maximal-correlation contraction constant from Lemma H.8 and Corollary H.9 in Appendix H.*

(iii) Gauge-invariant time-axis exponential clustering from the transfer-operator gap. *Fix $\delta \in (0, t/2)$ and a regulator* Reg. *In the KP corridor, the $L^2(\pi_{t,\mathrm{Reg}})$ contraction in* (ii) *gives a contraction factor $\rho(t, L) := q_W(t, L) \in (0, 1)$ for the mean-zero transfer operator T_t. Consequently, for all bounded gauge-invariant observables $\mathcal{O}, \mathcal{P} \in \mathfrak{Obs}^{\mathrm{lat}}_{t,\delta}$ (Definition 2.1) supported away from the time boundaries, the connected time-axis two-point functions satisfy an exponential clustering bound of the form (152) with rate parameter*

$$m_* \;\geq\; \frac{|\log q_W(t, L)|}{t}.$$

A self-contained proof is given in Proposition 11.9. This route does not *invoke the abstract sampler-to-observable transport assumption Assumption 12.6; see Remark 12.8.*

(iv) Thermodynamic limit $L \to \infty$ in the KP corridor. *Assume in addition that the coupling lies in an L-uniform strong-coupling corridor $0 < \beta \leq \beta_\star^\infty(t)$, where $\beta_\star^\infty(t) > 0$ depends only on (t, G) and is independent of L and of UV refinement (see Theorem I.1 in Section I). Then the spatial thermodynamic limit exists: the finite-volume slab Gibbs measures (with periodic spatial boundary conditions) converge as $L \to \infty$ in the local weak topology to a unique infinite-volume DLR state $\mu_{t,\infty,\mathrm{Reg}}$. The boundary marginals $\pi_{t,L,\mathrm{Reg}}$ converge on each fixed boundary window to a limiting boundary law $\pi_{t,\infty,\mathrm{Reg}}$, and the endpoint-disintegration transfer kernels converge on fixed windows to a limiting transfer specification $K_{t,\infty,\mathrm{Reg}}$. Moreover, the Euclidean-time clustering bound (152) holds for spatially local gauge-invariant cylinder observables (i.e. observables whose supporting edge set is contained in a fixed spatial window independent of L) with constants uniform in L (and uniform in UV refinement).*

Dependencies. Item (i) is proved in Appendix F (Appendix F). Item (ii) follows from the Wilson cross-slab bound (Lemma H.8 and Corollary H.9 in Appendix H) combined with the operator-theoretic contraction/mixing argument in Appendix G (Appendix G). Item (iii) is a consequence of the transfer gap from (ii) via Proposition 11.9. In particular, items (ii)–(iii) do not use the template semiconvexity/mLSI assumptions of Section 2.13 (Assumptions 2.19–2.21).

Proof. Item (i) is Corollary 2.29. Item (ii) is Corollary 2.25.

For (iii), combine the $L^2(\pi_{t,\mathrm{Reg}})$ contraction in (ii) with Proposition 11.9, which turns a mean-zero transfer-operator contraction for T_t into the gauge-invariant time-axis exponential clustering estimate (152) for observables supported away from the time boundaries. This transfer-side route does not invoke the sampler-to-observable transport assumption Assumption 12.6; see Remark 12.8.

Item (iv) is Theorem I.1 in Section I.

The auxiliary sampler-mixing result (Theorem 2.16) is logically independent and is *not* used in (iii). □

General template. Theorems 2.15 and 2.16 below record the abstract framework (a reusable template) for general slab Gibbs specifications. In contrast to Theorem 2.14, these results invoke the standing constructive inputs (notably stability/non-degeneracy and finite-range locality) that one expects to verify in concrete models when extending beyond the lattice corridor treated here.

2.10 Main structural estimates for the boundary potential

The first main theorem provides the regularity and growth controls on $U_{t,\mathrm{Reg}}$ needed to build Lyapunov drift and to control the low-mode dynamics.

Theorem 2.15 (Template track: local Hessian control and one-sided growth)**.** *Fix $t > 0$. Assume the standing structural inputs of Assumptions 2.9, 5.1 and 5.5 for the regulator family under consideration. For each $R > 0$ there exists a constant $C_2(t, R) > 0$ such that for every regulator tuple* Reg *the boundary potential $U_{t,\mathrm{Reg}}$ is twice Fréchet differentiable along Cameron–Martin directions on the set $\{x \in E : \|x\|_{\mathcal{H}} \le R\}$, and its $\mathcal{H}$-Hessian satisfies*

$$\sup_{\|x\|_{\mathcal{H}} \le R} \left\| D^2_{\mathcal{H}} U_{t,\mathrm{Reg}}(x) \right\|_{\mathrm{op}} \;\le\; C_2(t, R), \tag{6}$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm on $\mathcal{H}$. In particular, $\nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}$ is $C_2(t, R)$-Lipschitz (in $\mathcal{H}$) on $\{\|x\|_{\mathcal{H}} \le R\}$.

Moreover, there exist constants $K_1(t) \in (0,\infty)$ and $K_0(t) \in [0,\infty)$, depending on t (and on G and the chosen scheme) but independent of Reg, *such that for all $x \in E$,*

$$\left|\langle x, \nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}(x)\rangle_{\mathcal{H}}\right| \;\le\; K_1(t)\, \|x\|_{\mathcal{H}} + K_0(t). \tag{7}$$

A proof of the regulator-uniform choice of $(K_1(t), K_0(t))$ is given in Lemma 6.9.

2.11 Template track: quantitative mixing for the boundary Langevin sampler

The next theorem is the quantitative ergodicity statement for the boundary Langevin sampler (3).

Theorem 2.16 (Template track: regulator-uniform exponential mixing for the sampler)**.** *Fix $t > 0$. Assume the standing structural inputs of Assumptions 2.9, 5.1 and 5.5. There exist integers $m \ge 1$ and parameters $\eta \in (0, 1]$ and $s_\star > 0$, and constants $\lambda \in (0, 1)$ and $C_{\mathrm{mix}} \ge 1$, depending on t (and on G and the chosen regularisation scheme) but* independent of Reg, *such that the Markov semigroup $(P_s)_{s\ge0}$ of (3) satisfies:*

$$W^{(m)}_{1,\eta}\big(\delta_x P_{ns_\star},\, \delta_y P_{ns_\star}\big) \;\le\; C_{\mathrm{mix}}\, \lambda^n\, d_{m,\eta}(x, y), \qquad \forall x, y \in \mathcal{B}_{t,\mathrm{Reg}},\ \forall n \in \mathbb{N}. \tag{8}$$

In particular, for each Reg *there exists a unique invariant probability measure for (3), and it coincides with $\nu_{t,\mathrm{Reg}}$ from (1). Moreover,*

$$W^{(m)}_{1,\eta}\big(\delta_x P_{ns_\star},\, \nu_{t,\mathrm{Reg}}\big) \;\le\; C_{\mathrm{mix}}\, \lambda^n, \qquad \forall x \in \mathcal{B}_{t,\mathrm{Reg}},\ \forall n \in \mathbb{N}, \tag{9}$$

with constants uniform in Reg.

Proof location. *A regulator-uniform verification of the drift/minorisation hypotheses (avoiding any dimension blow-up) is given in Appendix E. Sections 7–8 present a fixed-*Reg *finite-dimensional version in a simpler metric for intuition.*

Remark 2.17 (What is (and is not) proved by Theorem 2.16). Theorem 2.16 is a statement about the *sampler* kernel $P_{s_\star}$ associated with (3). Euclidean-time statements (time-axis exponential clustering and transfer-operator spectral consequences) require corresponding information for the *Euclidean transfer kernel* $K_{t,\mathrm{Reg}}$ from Definition 2.7. We isolate the required bridge as a separate hypothesis below.

2.12 Bridge to Euclidean transfer: a comparison/identification hypothesis

Assumption 2.18 (Cylindrical transfer-kernel contraction)**.** Fix $t > 0$. There exist integers $m \geq 1$, parameters $\eta \in (0, 1]$, and constants $\lambda_K \in (0, 1)$ and $C_K \geq 1$, independent of Reg, such that the Euclidean transfer kernel $K_{t,\mathrm{Reg}}$ satisfies the *cylindrical* contraction estimate

$$W_{1,\eta}\Big((\Pi_m)_\# \delta_x K^n_{t,\mathrm{Reg}},\, (\Pi_m)_\# \delta_y K^n_{t,\mathrm{Reg}}\Big) \;\leq\; C_K\, \lambda_K^n, \qquad \forall x, y \in \mathcal{B}_{t,\mathrm{Reg}},\ \forall n \in \mathbb{N}, \tag{10}$$

where $W_{1,\eta}$ denotes the Wasserstein distance on the finite-dimensional range of Π_m associated with the bounded cost $d_\eta(u, v) := 1 \wedge \eta\, d_{\mathrm{proj}}(u, v)$, where d_{proj} is any fixed distance on the (finite-dimensional) range of Π_m (in the Wilson track: product geodesic on G^W). Equivalently, (10) yields exponential mixing for Π_m-cylindrical observables with bounded d_η-Lipschitz seminorm.

This hypothesis may be verified either (i) directly for $K_{t,\mathrm{Reg}}$ (preferred), or (ii) via a comparison principle that transfers the sampler contraction (8) to $K_{t,\mathrm{Reg}}$.

2.13 A stronger transfer bridge via log–Sobolev and modified log–Sobolev inequalities

For applications to OS reconstruction, it is often preferable to control the transfer kernel directly in an L^2-based Dirichlet form, rather than via a sampler-to-transfer comparison. The following "LSI $\Rightarrow$ mLSI" mechanism is the transfer-side analogue of the weak Harris route.

Assumption 2.19 (Semiconvexity on the mean–zero boundary sector)**.** Fix $t > 0$ and let Π_0 denote the orthogonal projection onto the mean–zero sector $\mathsf{H}_0 := \Pi_0 \mathsf{H}$ (cf. Section 2). There exists a constant $\kappa = \kappa(t) \geq 0$, independent of the UV cutoff and auxiliary truncations (at fixed (t, L)), such that for all x and all $h \in \mathsf{H}_0$,

$$\langle h,\, D^2_{\mathsf{H}} W_{t,\mathrm{Reg}}(x)\, h\rangle_{\mathsf{H}} \;\geq\; -\kappa\, \|h\|^2_{\mathsf{H}}, \qquad \text{and} \qquad \kappa < \frac{2}{t}. \tag{11}$$

Remark 2.20 (Template interpretation of the bound $\kappa < 2/t$)*.* In the *optional Euclidean-reference template* on a linear boundary space $(E, \mathsf{H}_0, \mu_{0,t,\mathrm{Reg}})$, one may view the "free bridge" contribution as providing a strictly positive curvature margin of size $2/t$ on the mean–zero sector H_0, while the interaction contributes at worst $-\kappa$ in Assumption 2.19. The residual margin $m(t) := 2/t - \kappa > 0$ is then the quantitative convexity that drives the log–Sobolev and contraction mechanisms.

In the *Wilson lattice track*, no Gaussian bridge decomposition is invoked. Here the same inequality $\kappa < 2/t$ is used only as a convenient way to record a strictly positive semiconvexity margin on the mean–zero sector for the disintegration-defined boundary potential; the needed lower-Hessian and response bounds are verified directly in the KP/cluster-expansion corridor (Appendix H) using intrinsic left/right–invariant derivatives on the compact group.

Assumption 2.21 (Deterministic gradient contraction for the transfer)**.** Fix $t > 0$. There exists $q = q(t, L) \in [0, 1)$, independent of the UV cutoff and auxiliary truncations, such that for all cylindrical f,

$$\|\nabla(K_{t,\mathrm{Reg}} f)(x)\|_{\mathsf{H}} \;\leq\; q\, K_{t,\mathrm{Reg}}(\|\nabla f\|_{\mathsf{H}})(x), \qquad \text{for } \nu^{(\mathrm{Reg})}\text{-a.e. } x, \tag{12}$$

where $\nu^{(\mathrm{Reg})}$ is the invariant one–boundary law of $K_{t,\mathrm{Reg}}$ and ∇ denotes the H-gradient.

Remark 2.22 (Template-only: semiconvexity/mLSI is not used in the Wilson corridor proof). This semiconvexity/mLSI mechanism belongs to the *template track* and is *not* used in the Wilson-lattice proofs of Theorem 2.14(ii)–(iii). Those items instead follow from the Wilson cross-slab polymer correlation bound in Appendix H (Lemma H.8 and Corollary H.9) together with the corridor argument in Appendix G.

Theorem 2.23 (Uniform mLSI and L^2 mixing for the slab transfer)**.** *Assume Assumptions 2.19 and 2.21. Then the invariant law $\nu^{(\mathrm{Reg})}$ satisfies a Gross log–Sobolev inequality on H_0 with constant*

$$\alpha_\nu(t) \;\geq\; \frac{2}{t} - \kappa \;>0 \qquad \text{(uniform in the UV cutoff and auxiliary truncations),} \tag{13}$$

and the reversible kernel $K_{t,\mathrm{Reg}}$ satisfies a modified log–Sobolev inequality (mLSI) on H_0 with constant

$$\rho(t) \;:=\; \frac{\alpha_\nu(t)\,(1-q^2)}{4} \;>0, \tag{14}$$

uniform in the UV cutoff and auxiliary truncations. In particular, the transfer operator on $L^2(\nu^{(\mathrm{Reg})})$ has spectral gap at least $\rho(t)$ on H_0 and

$$\|K^n_{t,\mathrm{Reg}}f - \nu^{(\mathrm{Reg})}(f)\|_{L^2(\nu^{(\mathrm{Reg})})} \;\leq\; e^{-\rho(t)\,n}\,\|f - \nu^{(\mathrm{Reg})}(f)\|_{L^2(\nu^{(\mathrm{Reg})})}, \qquad \forall n \in \mathbb{N}. \tag{15}$$

Remark 2.24 (A corridor verification for the transfer–side bridge). Appendix G proves an explicit small-β / strong-coupling *KP corridor* for a concrete finite-range Wilson lattice regulator family (compact gauge group G, fixed slab thickness $t > 0$ and fixed spatial size L). In particular, the crossing-polymer bound of Lemma H.8 (Appendix H) implies the transfer operator norm contraction (211) with a Wilson coefficient $q_W(t, L) < 1$ (Corollary H.9), hence geometric L^2 mixing and a positive transfer spectral gap in the sense of (212)–(213).

Corollary 2.25 (Transfer-side L^2 mixing in a KP corridor)**.** *Fix $t > 0$ and a spatial torus size L, and consider the regulator family of Appendix G. Then there exists $\beta_0 = \beta_0(t, L, G) > 0$ such that for $0 < \beta \leq \beta_0$ the slab transfer kernel $K_{t,\mathrm{Reg}}$ satisfies the Wilson-intrinsic $L^2(\pi_{t,\mathrm{Reg}})$ mixing estimate (212) with contraction coefficient $q_W(t, L) < 1$ (uniform in the UV cutoff and auxiliary truncations at fixed (t, L)), and in particular the L^2 transfer spectral gap bound (213).*

Remark 2.26 (What “small β” means in the corridor). In the KP corridor the small parameter is the inverse temperature β (equivalently strong coupling) in a fixed gauge chart (a high-temperature / cluster-expansion regime at fixed (t, L)), used to control polymer activities and their derivatives uniformly in the UV refinement. This is *not* the continuum-limit weak-coupling scaling $g = g(\Lambda) \to 0$ associated with asymptotic freedom. Compatibility of a corridor analysis with a continuum limit would require an additional renormalisation input that is outside the scope of this paper.

Remark 2.27 (A toy case where Assumption 2.18 holds). In the free Gaussian case $U_{t,\mathrm{Reg}} \equiv 0$ (so $\nu_{t,\mathrm{Reg}} = \mu_{0,t,\mathrm{Reg}}$), the Euclidean transfer kernel $K_{t,\mathrm{Reg}}$ is Gaussian and coincides with the time-t transition kernel of the associated Ornstein–Uhlenbeck trace process. In this setting, the sampler dynamics (3) is itself an Ornstein–Uhlenbeck process, and one verifies (10) explicitly (indeed, with a sharp rate coming from the OU spectrum). This provides a consistency check: the “bridge” hypothesis is automatic in the linear/Gaussian regime, while in interacting models it amounts to proving either that $K_{t,\mathrm{Reg}}$ is the time-t kernel of a Markov dynamics comparable to (3), or that a comparison principle transfers contraction from a convenient sampler to the true slab transfer kernel.

Proposition 2.28 (Concrete verification in a finite-range lattice regulator)**.** *There exists a concrete finite-range lattice regulator family in which Assumption 2.18 can be verified* within this paper*. Namely, for a finite-range lattice regularisation of the slab with compact gauge group G, a Wilson-type plaquette action (or any uniformly bounded local action with a uniformly bounded number of interaction terms per edge), and the corresponding one-step Euclidean transfer update (the endpoint-disintegration kernel $K_{t,\mathrm{Reg}}$) restricted to a fixed unit spatial window W, one obtains a uniform local Doeblin minorisation on a fixed finite window. This yields the geometric contraction* (10) *for a suitable choice of (m, η) that is independent of the regulator tuple. A self-contained proof is given in Appendix F.*

Corollary 2.29 (Cylindrical transfer-kernel contraction in the finite-range lattice family)**.** *In the finite-range lattice regulator family described in Appendix F, Assumption 2.18 holds (with constants uniform in the UV refinement at fixed (t, L)). Consequently, within this lattice family the transfer-kernel consequences that are stated under Assumption 2.18 apply without invoking any additional identification/comparison hypothesis between the transfer kernel and the auxiliary boundary dynamics (still subject to the separate response/density assumptions where stated).*

2.14 Dependency map and standing assumptions

For the reader's convenience, we summarise the logical dependencies among the main statements.

- **Route A (template sampler ⇒ slab observables).** Uses transfer-kernel mixing (either via Assumption 2.18 or via Theorem 2.23), together with the response-moment input Assumption 12.6 and the density hypothesis Assumption 13.11, to obtain template Schwinger-function clustering and (conditional) spectral consequences.
- **Route B (Wilson/KP corridor: transfer-gap route).** Bypasses transport entirely and deduces time-axis clustering for lattice observables in $\mathfrak{Obs}^{\mathrm{lat}}_{t,\delta}$ directly from the $L^2(\pi_{t,\mathrm{Reg}})$ contraction/spectral gap of the disintegration-defined transfer operator (Theorem 2.14(ii)–(iii) and Proposition 11.9), with constants uniform under UV refinement and persistence under $L \to \infty$ in the corridor (Theorem I.1).

All results in this paper are at fixed slab thickness $t > 0$, and (unless explicitly stated otherwise) at fixed (t, L).

2.15 Time-axis exponential clustering in the OS limit (conditional)

Remark 2.30 (On the use of the term time-axis exponential clustering in this paper)*.* We use "time-axis exponential clustering" in a *restricted, slab/time-axis* sense: exponential clustering for observables separated in the Euclidean time direction, as obtained from slab concatenation/transfer-operator estimates. We do not claim verification of the full Osterwalder–Schrader axiom list on $\mathbb{R}^4$ in this manuscript.

To formulate time-axis exponential clustering, we consider a limit along regulators $\mathrm{Reg} \to \infty$ (e.g. $L \to \infty$, $\Lambda \to \infty$). Existence of subsequential limits depends on the chosen regularisation scheme; here we isolate what we need.

Assumption 2.31 (Existence of an OS limit)**.** For each fixed $t > 0$ there exists a sequence of regulators $\mathrm{Reg}_k \to \infty$ along which the regulated slab measures $\mu^{\mathrm{bulk}}_{t,\mathrm{Reg}_k}$ converge (in the sense of finite-dimensional distributions on local observables) to a reflection-positive OS limit measure μ_t on fields over $S_t = [0, t] \times \mathbb{R}^3$.

Corollary 2.32 (OS4 exponential clustering at fixed $t > 0$ (conditional))**.** *Assume Assumptions 2.31 and 2.18. Fix $t > 0$. There exist constants $C_{\text{clust}}(t) \geq 1$ and $\rho(t) > 0$ such that for any pair of bounded gauge-invariant local observables F, G supported in time slabs separated by a gap of size $s > 0$, one has*

$$\left|\mathrm{Cov}_{\mu_t}(F, \tau_s G)\right| \;\leq\; C_{\text{clust}}(t)\, \mathrm{e}^{-\rho(t)\, s}\; \|F\|_{L^\infty(\mu_t)}\, \|G\|_{L^\infty(\mu_t)}\,. \tag{16}$$

2.16 Spectral gap consequence (conditional)

time-axis exponential clustering yields a spectral-gap lower bound for the Euclidean-time transfer semigroup associated with $(\mu_t)_{t>0}$ in the OS reconstruction.

Corollary 2.33 (Spectral gap for the Euclidean-time transfer semigroup (conditional))**.** *Assume Assumptions 2.31 and 2.18. Fix $t > 0$ and let $(\mathcal{K}, \Omega, T_s)_{s\geq 0}$ be the OS-reconstructed Hilbert space, vacuum vector, and Euclidean-time translation semigroup associated with the time direction on S_t. Let Π_0 denote the orthogonal projection onto* $\mathrm{span}\{\Omega\}$*. Then, on the closed subspace generated by gauge-invariant local observables, there exists $\rho(t) > 0$ such that*

$$\|T_s - \Pi_0\| \;\leq\; C\, \mathrm{e}^{-\rho(t)s}, \qquad s \geq 0, \tag{17}$$

for some finite constant $C \geq 1$. Equivalently, if $T_s = \mathrm{e}^{-sH}$ for a self-adjoint generator $H \geq 0$, then

$$\inf\bigl(\mathrm{spec}(H) \setminus \{0\}\bigr) \;\geq\; \rho(t) \tag{18}$$

on the same gauge-invariant subspace.

Remark 2.34 (Fixed-t nature of the gap)*.* Corollary 2.33 is a fixed-t statement: $\rho(t) > 0$ for each $t > 0$. Understanding the dependence on t (e.g. as $t \to \infty$) is a separate problem and is not pursued here.

2.17 Constants ledger (dependencies and uniformities)

For later use, we record the key constants and what they may depend on.

Constant / object	**May depend on**
$C_2(t, R)$ in (6)	t, R, G, choice of regularisation scheme
$K_1(t), K_0(t)$ in (7)	t, G, regularisation scheme
m (low-mode dimension)	t, G, regularisation scheme (chosen once, fixed in Reg)
η in $d_{m,\eta}$	t, G (chosen once, fixed in Reg)
$s_\star$ (sampler skeleton time)	t (chosen once, fixed in Reg)
λ, C_{mix} in (8)	t, G, regularisation scheme
λ_K, C_K in (10)	t, G, regularisation scheme
$C_{\text{clust}}(t), \rho(t)$ in (16)	t, G, regularisation scheme
Uniform in	the UV cutoff and auxiliary truncations at fixed (t, L)

Remark 2.35 (Operational meaning of "uniform in regulators")*.* In proofs, "uniform in regulators" means that the bounds entering the drift/minorisation hypotheses and the derivative/Hessian controls are established with constants independent of the UV cutoff and auxiliary truncations (i.e. of $\mathrm{Reg}_{\mathrm{UV}}$) at fixed (t, L). When OS limits along $\mathrm{Reg} \to \infty$ are considered, the conditional Euclidean-time conclusions additionally require uniform contraction for the transfer kernel $K_{t,\mathrm{Reg}}$ as in Assumption 2.18.

3 Differentiation of the boundary potential

This section isolates the analytic mechanism behind Theorem 2.15: we express first and second Cameron–Martin derivatives of the boundary potential $U_{t,\mathrm{Reg}}$ in terms of expectations and covariances under a bulk measure conditioned on boundary data. The point is bookkeeping: once these identities are in place, the proof of uniform C^2-bounds reduces to (i) quasi-locality of the bulk-to-boundary response and (ii) uniform moment bounds.

3.1 Cameron–Martin derivatives on an abstract Wiener space

Let $(E, \mathcal{H}, \mu_0)$ be an abstract Wiener space as in Section 2.4. For a measurable function $f : E \to \mathbb{R}$ and $h \in \mathcal{H}$, define the directional derivative (when it exists)

$$D_h f(x) \;:=\; \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon}. \tag{19}$$

If $D_h f(x)$ exists for all $h \in \mathcal{H}$ and is continuous and linear in h, we write $\nabla_{\mathcal{H}} f(x) \in \mathcal{H}$ for the Cameron–Martin gradient, defined by

$$D_h f(x) = \langle \nabla_{\mathcal{H}} f(x), h \rangle_{\mathcal{H}}\,, \qquad h \in \mathcal{H}. \tag{20}$$

Similarly, if second derivatives exist and define a bounded bilinear form on $\mathcal{H} \times \mathcal{H}$, we write $D^2_{\mathcal{H}} f(x)$ for the corresponding bounded self-adjoint operator on $\mathcal{H}$.

3.2 Boundary partition function as a log-Laplace functional

Fix $t > 0$ and regulators Reg. In concrete constructions, the boundary density in (1) arises from integrating out bulk fields conditioned on boundary data. Abstractly, we may write (locally, after choosing a convenient bulk reference law) the normalizing factor as

$$Z_{t,\mathrm{Reg}}(x) \;=\; \mathbb{E}\big[\exp\{-\mathcal{V}_{t,\mathrm{Reg}}(x; \Xi)\}\big], \tag{21}$$

where Ξ denotes a (regulated) bulk random field under a reference probability law $\mathbb{P}$ and $\mathcal{V}_{t,\mathrm{Reg}}(x; \Xi)$ is a bulk functional depending on the boundary datum x. The boundary potential can then be written as

$$U_{t,\mathrm{Reg}}(x) \;=\; -\log Z_{t,\mathrm{Reg}}(x) + \log Z_{t,\mathrm{Reg}}(0), \tag{22}$$

which is compatible with the density representation (1) after fixing additive constants.

It is convenient to introduce the tilted probability measure $\mathbb{P}_x$ on the bulk configuration space, defined by

$$\frac{\mathrm{d}\mathbb{P}_x}{\mathrm{d}\mathbb{P}}(\Xi) \;:=\; \frac{\exp\{-\mathcal{V}_{t,\mathrm{Reg}}(x; \Xi)\}}{Z_{t,\mathrm{Reg}}(x)}. \tag{23}$$

We write $\mathbb{E}_x[\cdot]$ for expectation under $\mathbb{P}_x$.

3.3 First and second derivative identities

The following lemma is a standard differentiation-under-the-integral statement for log-partition functions, recorded here in the precise form needed for uniform bounds.

Lemma 3.1 (Log-partition derivatives)**.** *Fix $t > 0$ and* Reg. *Assume that for each $h, k \in \mathcal{H}$:*

(i) *for $\mathbb{P}$-a.e. Ξ, the map $x \mapsto \mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)$ is twice differentiable along $\mathcal{H}$-directions, with directional derivatives $D_h\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)$ and $D^2_{h,k}\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)$;*

(ii) *there exists a neighbourhood $\mathcal{U}$ of x in $\mathcal{H}$ such that*

$$\sup_{x'\in\mathcal{U}} \mathbb{E}\Big[\exp\{-\mathcal{V}(x';\Xi)\}\big(|D_h\mathcal{V}(x';\Xi)| + |D^2_{h,k}\mathcal{V}(x';\Xi)| + |D_h\mathcal{V}(x';\Xi)|^2\big)\Big] < \infty.$$

Then $Z_{t,\mathrm{Reg}}(x)$ is twice differentiable along $\mathcal{H}$-directions and

$$D_h \log Z_{t,\mathrm{Reg}}(x) = -\,\mathbb{E}_x[D_h\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)], \tag{24}$$

$$D^2_{h,k} \log Z_{t,\mathrm{Reg}}(x) = -\,\mathbb{E}_x[D^2_{h,k}\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)] + \mathrm{Cov}_x\big(D_h\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi),\, D_k\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)\big). \tag{25}$$

Consequently, the boundary potential $U_{t,\mathrm{Reg}}(x)$ satisfies

$$D_h U_{t,\mathrm{Reg}}(x) = \mathbb{E}_x[D_h\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)], \tag{26}$$

$$D^2_{h,k} U_{t,\mathrm{Reg}}(x) = \mathbb{E}_x[D^2_{h,k}\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)] - \mathrm{Cov}_x\big(D_h\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi),\, D_k\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)\big). \tag{27}$$

Remark 3.2 (What must be controlled). The operator bound (6) for $D^2_{\mathcal{H}}U_{t,\mathrm{Reg}}(x)$ will follow from (27) once we prove uniform bounds, for $\|x\|_{\mathcal{H}} \le R$, on:

$$\sup_{\|h\|_{\mathcal{H}}=\|k\|_{\mathcal{H}}=1} \Big|\mathbb{E}_x[D^2_{h,k}\mathcal{V}(x;\Xi)]\Big| \quad \text{and} \quad \sup_{\|h\|_{\mathcal{H}}=1} \mathrm{Var}_x\big(D_h\mathcal{V}(x;\Xi)\big), \tag{28}$$

with constants independent of Reg. Similarly, the one-sided growth bound (7) will follow from (26) with $h = x$ together with a uniform control on $\mathbb{E}_x[D_x\mathcal{V}(x;\Xi)]$.

3.4 Reduction to quasi-local response and uniform moments

The derivative identities above are exact; the work is to bound the terms uniformly in regulators. The remainder of the paper implements the following plan.

R1. Quasi-local response. We show that $D_h\mathcal{V}(x;\Xi)$ and $D^2_{h,k}\mathcal{V}(x;\Xi)$ can be expressed as integrals of local (or quasi-local) densities against h and (h,k), respectively, with kernels that decay away from the boundary support of h. This step is specific to the slab geometry and the regulated Yang–Mills action.

R2. Uniform integrability under $\mathbb{P}_x$. We prove moment/exponential-moment bounds for the local densities entering the response, uniformly in Reg and locally uniformly in x (typically on $\{\|x\|_{\mathcal{H}} \le R\}$). Combined with quasi-locality, this yields the uniform bounds in (28).

To keep the presentation modular, we implement **R1** in Section 4 and **R2** in Section 5. These two inputs jointly yield Theorem 2.15 in Section 6.

4 Optional template: Quasi-local response kernels for the slab boundary law

Template-only notice. This section develops a linear/Gaussian bookkeeping template (quasi-local response kernels, harmonic extension, Dirichlet-to-Neumann operators) on an optional additive boundary realisation. It is *not* used in the Wilson lattice track, where boundary variables live on the compact manifold $\mathsf{B}_\partial = G^{E_\partial}$ and the corridor/Wilson transfer estimates are obtained directly from KP/cluster expansion outputs (Appendix H).

This section fixes a concrete regulator chart for the slab and derives explicit formulas for the first and second Cameron–Martin derivatives of the bulk functional entering the log-partition representation of Section 3. The only structural point we need later is:

> *All dependence on boundary data enters through a* <u>linear</u> *harmonic extension map, and the functional derivatives of the regulated interaction are* <u>finite-range local</u> *at scale* Λ^{-1}.

Sampler vs. transfer kernel. The response formulas below are written for a boundary datum at time 0. They are compatible with the Euclidean transfer-kernel viewpoint (Definition 2.7) because the same harmonic-extension and finite-range locality statements hold *with two boundary traces* (x_-, x_+). For clarity we record both the one-sided and two-sided harmonic extension/DN operators; the locality proofs are identical.

4.1 Field spaces, Fourier conventions, and the UV smoothing

Fix $L > 0$ and write $\mathbb{T}_L^3 = (\mathbb{R}/L\mathbb{Z})^3$ with coordinates $x \in [0, L)^3$. Let $S_{t,L} = [0, t] \times \mathbb{T}_L^3$ and denote points by $X = (x_0, x)$. We consider $\mathfrak{g}$-valued one-forms $A = (A_\mu)_{\mu=0}^3$ on $S_{t,L}$, with each component A_μ a $\mathfrak{g}$-valued function on $S_{t,L}$.

To keep all objects finite-dimensional and classically differentiable, we impose a spatial Fourier cutoff. Let

$$\Gamma_L := \frac{2\pi}{L}\mathbb{Z}^3, \qquad \Gamma_{L,\Lambda} := \{p \in \Gamma_L : \ |p|_\infty \le \Lambda\}.$$

For a (vector-valued) function f on $\mathbb{T}_L^3$, write

$$f(x) = \sum_{p \in \Gamma_L} \widehat{f}(p)\, e^{ip\cdot x}, \qquad \widehat{f}(p) = \frac{1}{L^3} \int_{\mathbb{T}_L^3} f(x)\, e^{-ip\cdot x}\, \mathrm{d}x.$$

Define the spatial projector P_Λ by

$$\mathsf{P}_\Lambda f \ = \sum_{p \in \Gamma_{L,\Lambda}} \widehat{f}(p)\, e^{ip\cdot x}.$$

All fields and test functions below are assumed to lie in the range of P_Λ (componentwise), so they are smooth and finite-dimensional in space.

We also introduce a *finite-range* UV smoothing in spacetime. Fix once and for all a nonnegative mollifier $\rho \in C_c^\infty(\mathbb{R}^4)$ with $\int_{\mathbb{R}^4} \rho = 1$ and $\mathrm{supp}(\rho) \subset B(0, 1)$ (unit Euclidean ball). For $\Lambda \ge 1$ set $\rho_\Lambda(X) = \Lambda^4 \rho(\Lambda X)$ and define

$$(\mathsf{S}_\Lambda f)(X) := (\rho_\Lambda * f)(X) = \int_{\mathbb{R}\times\mathbb{R}^3} \rho_\Lambda(X - Y)\, f(Y)\, \mathrm{d}Y,$$

where f is extended periodically in x and by reflection outside $[0, t]$ in the time coordinate (so that S_Λ is well-defined on the slab and preserves boundary conditions). By construction,

$$\mathrm{supp}(\rho_\Lambda) \subset B(0, \Lambda^{-1}) \quad \Longrightarrow \quad (\mathsf{S}_\Lambda f)(X) \text{ depends only on } f \text{ on } B(X, \Lambda^{-1}). \tag{29}$$

The projector P_Λ and the smoother S_Λ commute on spatially truncated fields (and we may silently replace S_Λ by $\mathsf{P}_\Lambda \mathsf{S}_\Lambda$ to remain in the finite-dimensional space).

4.2 Free slab action and harmonic extensions

We define the free (Gaussian) reference structure using the Dirichlet Laplacian on the slab. Let $\Delta = \partial_0^2 + \Delta_x$ denote the Euclidean Laplacian. For each component A_μ, define the free action

$$S^0_{t,L,\Lambda}(A) \;\coloneqq\; \frac{1}{2}\sum_{\mu=0}^{3}\int_{S_{t,L}} |\nabla A_\mu(X)|^2 \,\mathrm{d}X, \tag{30}$$

on the affine space of spatially truncated fields with prescribed endpoint traces. We use the boundary space

$$\mathcal{B}_{L,\Lambda} \coloneqq \Big\{x : \mathbb{T}^3_L \to \mathfrak{g}^{\oplus 4} \text{ spatially truncated}\Big\},$$

and we write $x_-, x_+ \in \mathcal{B}_{L,\Lambda}$ for the boundary traces at time 0 and t, respectively. The homogeneous fluctuation space is

$$\mathcal{X}^0_{t,L,\Lambda} \coloneqq \Big\{\zeta :\; S_{t,L} \to \mathfrak{g}^{\oplus 4} \text{ spatially truncated},\; \zeta(0,\cdot) = \zeta(t,\cdot) = 0\Big\}.$$

On $\mathcal{X}^0_{t,L,\Lambda}$, the quadratic form (30) defines a centered Gaussian measure.

Two-sided harmonic extension. The harmonic extension operator from endpoint data is defined componentwise.

Definition 4.1 (Two-sided harmonic extension)**.** Given $(x_-, x_+) \in \mathcal{B}_{L,\Lambda} \times \mathcal{B}_{L,\Lambda}$, define $u = \mathsf{H}_t(x_-, x_+)$ as the unique spatially truncated solution of

$$\begin{cases} -\Delta u = 0 & \text{on } (0,t) \times \mathbb{T}^3_L, \\ u(0,\cdot) = x_-, \qquad u(t,\cdot) = x_+. \end{cases} \tag{31}$$

We also write $\mathsf{H}^-_t x \coloneqq \mathsf{H}_t(x, 0)$ and $\mathsf{H}^+_t x \coloneqq \mathsf{H}_t(0, x)$, so that $\mathsf{H}_t(x_-, x_+) = \mathsf{H}^-_t x_- + \mathsf{H}^+_t x_+$.

Lemma 4.2 (Explicit Fourier formula and maximum principle)**.** *Let $(x_-, x_+) \in \mathcal{B}_{L,\Lambda} \times \mathcal{B}_{L,\Lambda}$ and write $x_\pm(x) = \sum_{p\in\Gamma_{L,\Lambda}} \widehat{x}_\pm(p)e^{ip\cdot x}$. Then $u = \mathsf{H}_t(x_-, x_+)$ is given by*

$$u(x_0, x) = \sum_{p\in\Gamma_{L,\Lambda}} \left[\widehat{x}_-(p)\,\frac{\sinh(|p|(t-x_0))}{\sinh(|p|t)} + \widehat{x}_+(p)\,\frac{\sinh(|p|x_0)}{\sinh(|p|t)}\right] e^{ip\cdot x}, \qquad (|p| \coloneqq \sqrt{p_1^2 + p_2^2 + p_3^2}), \tag{32}$$

with the convention that for $p = 0$ the ratios are interpreted as limits:

$$\frac{\sinh(|p|(t-x_0))}{\sinh(|p|t)}\Big|_{p=0} = 1 - \frac{x_0}{t}, \qquad \frac{\sinh(|p|x_0)}{\sinh(|p|t)}\Big|_{p=0} = \frac{x_0}{t}.$$

Moreover, H_t is a contraction in L^∞:

$$\sup_{(x_0,x)\in S_{t,L}} |u(x_0, x)| \le \max\Big\{\sup_{x\in\mathbb{T}^3_L} |x_-(x)|,\; \sup_{x\in\mathbb{T}^3_L} |x_+(x)|\Big\}. \tag{33}$$

Proof. Fix $p \in \Gamma_{L,\Lambda}$ and consider the mode $u_p(x_0)e^{ip\cdot x}$. Solving $-u_p''(x_0) + |p|^2 u_p(x_0) = 0$ with endpoint values $u_p(0) = \widehat{x}_-(p)$ and $u_p(t) = \widehat{x}_+(p)$ yields (32). The L^∞ bound follows from the maximum principle for each scalar component on the cylinder. $\square$

4.3 Dirichlet-to-Neumann forms (one-sided and two-sided)

The boundary Gaussian structure is governed by Dirichlet-to-Neumann operators.

Definition 4.3 (Two-sided Dirichlet-to-Neumann map)**.** Define the operator $\mathsf{N}^{\mathrm{end}}_{t,L,\Lambda}$ on $\mathcal{B}_{L,\Lambda} \times \mathcal{B}_{L,\Lambda}$ by

$$\mathsf{N}^{\mathrm{end}}_{t,L,\Lambda}(x_-, x_+) \;:=\; \Big(-\partial_0 u\big|_{x_0=0},\quad +\partial_0 u\big|_{x_0=t}\Big), \qquad u = \mathsf{H}_t(x_-, x_+). \tag{34}$$

Equivalently, $\mathsf{N}^{\mathrm{end}}_{t,L,\Lambda}$ is the 2×2 block operator

$$\mathsf{N}^{\mathrm{end}}_{t,L,\Lambda} = \begin{pmatrix} \mathsf{N}_{--} & \mathsf{N}_{-+} \\ \mathsf{N}_{+-} & \mathsf{N}_{++} \end{pmatrix}$$

with $\mathsf{N}_{--}x = -\partial_0(\mathsf{H}^-_t x)|_0$, $\mathsf{N}_{++}x = +\partial_0(\mathsf{H}^+_t x)|_t$, and the off-diagonal blocks $\mathsf{N}_{-+}x = -\partial_0(\mathsf{H}^+_t x)|_0$, $\mathsf{N}_{+-}x = +\partial_0(\mathsf{H}^-_t x)|_t$.

Definition 4.4 (One-sided Dirichlet-to-Neumann operator)**.** Define $\mathsf{N}_{t,L,\Lambda} : \mathcal{B}_{L,\Lambda} \to \mathcal{B}_{L,\Lambda}$ by

$$\mathsf{N}_{t,L,\Lambda} x \;:=\; -\partial_0(\mathsf{H}^-_t x)\big|_{x_0=0}. \tag{35}$$

This is the special case of (34) with $x_+ = 0$.

Lemma 4.5 (Fourier symbols)**.** *The one-sided operator $\mathsf{N}_{t,L,\Lambda}$ is self-adjoint and positive definite on $\mathcal{B}_{L,\Lambda}$ with Fourier multiplier*

$$(\widehat{\mathsf{N}_{t,L,\Lambda} x})(p) = \omega_t(p)\, \widehat{x}(p), \qquad \omega_t(p) := \begin{cases} |p| \coth(|p|t), & p \neq 0, \\ t^{-1}, & p = 0. \end{cases} \tag{36}$$

For the two-sided map, $\mathsf{N}^{\mathrm{end}}_{t,L,\Lambda}$ has the 2×2 Fourier-multiplier matrix

$$(\mathsf{N}^{\mathrm{end}}_{t,L,\Lambda} \widehat{(x_-, x_+)})(p) = |p| \begin{pmatrix} \coth(|p|t) & -\operatorname{csch}(|p|t) \\ -\operatorname{csch}(|p|t) & \coth(|p|t) \end{pmatrix} \begin{pmatrix} \widehat{x}_-(p) \\ \widehat{x}_+(p) \end{pmatrix} \qquad (p \neq 0), \tag{37}$$

and for $p = 0$ the matrix is $t^{-1}\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$ in the obvious sense.

Proof. Differentiate (32) in x_0 and evaluate at $x_0 = 0$ and $x_0 = t$. For the one-sided case, set $x_+ = 0$. $\square$

Lemma 4.6 (Energy identity)**.** *Let $u = \mathsf{H}_t(x_-, x_+)$. Then*

$$\int_{S_{t,L}} |\nabla u(X)|^2 \, \mathrm{d}X = \langle x_-, -\partial_0 u|_0 \rangle_{L^2(\mathbb{T}^3_L)} + \langle x_+, +\partial_0 u|_t \rangle_{L^2(\mathbb{T}^3_L)} = \Big\langle (x_-, x_+), \mathsf{N}^{\mathrm{end}}_{t,L,\Lambda}(x_-, x_+) \Big\rangle_{L^2 \oplus L^2}. \tag{38}$$

In particular, taking $x_+ = 0$ recovers the one-sided identity

$$\int_{S_{t,L}} |\nabla(\mathsf{H}^-_t x)(X)|^2 \, \mathrm{d}X = \langle x, \mathsf{N}_{t,L,\Lambda} x \rangle_{L^2(\mathbb{T}^3_L)}. \tag{39}$$

Proof. Integrate by parts using $-\Delta u = 0$ and note that the outward normal derivative at $x_0 = 0$ is $-\partial_0$ and at $x_0 = t$ is $+\partial_0$. $\square$

4.4 Gaussian conditioning: bulk fluctuations and induced endpoint/boundary laws

Let $\gamma^D := \gamma^D_{t,L,\Lambda}$ denote the centered Gaussian measure on $\mathcal{X}^0_{t,L,\Lambda}$ with density proportional to $\exp(-S^0_{t,L,\Lambda}(\zeta))$ with respect to Lebesgue measure on the finite-dimensional space $\mathcal{X}^0_{t,L,\Lambda}$. For endpoint data (x_-, x_+) define the affine bulk field

$$A^{(x_-,x_+)} \;:=\; \mathsf{H}_t(x_-, x_+) + \zeta, \qquad \zeta \sim \gamma^D_{t,L,\Lambda}.$$

Lemma 4.7 (Induced free endpoint law)**.** *There exists a (normalised) Gaussian measure $\mu^{\mathrm{end}}_{0,t,L,\Lambda}$ on $\mathcal{B}_{L,\Lambda} \times \mathcal{B}_{L,\Lambda}$ such that for every bounded measurable Ψ,*

$$\int \Psi(x_-, x_+)\, \mu^{\mathrm{end}}_{0,t,L,\Lambda}(\,\mathrm{d}x_-,\, \mathrm{d}x_+) = \frac{1}{Z^{0,\mathrm{end}}_{t,L,\Lambda}} \iint \Psi(x_-, x_+)\, \exp\Big(-S^0_{t,L,\Lambda}(\mathsf{H}_t(x_-, x_+)+\zeta)\Big)\, \mathrm{d}\zeta\, \mathrm{d}x_-\, \mathrm{d}x_+, \tag{40}$$

and $\mu^{\mathrm{end}}_{0,t,L,\Lambda}$ has density proportional to

$$\exp\Big(-\frac{1}{2}\, \Big\langle (x_-, x_+), \mathsf{N}^{\mathrm{end}}_{t,L,\Lambda}(x_-, x_+) \Big\rangle_{L^2\oplus L^2}\Big) \tag{41}$$

with respect to Lebesgue measure on $\mathcal{B}_{L,\Lambda} \times \mathcal{B}_{L,\Lambda}$.

Proof. As in the one-sided case, use the orthogonal decomposition induced by harmonic extension plus Dirichlet fluctuations: the cross term vanishes because ζ has homogeneous Dirichlet boundary and $-\Delta \mathsf{H}_t(x_-, x_+) = 0$. Then apply Lemma 4.6 to identify the boundary quadratic form. □

Remark 4.8 (Connection to the transfer kernel)*.* The free Euclidean transfer kernel is the disintegration of $\mu^{\mathrm{end}}_{0,t,L,\Lambda}$ with respect to its first marginal, exactly as in Definition 2.7. We do not need its explicit Gaussian conditional form here; what matters for this section is that endpoint dependence enters through $\mathsf{H}_t(x_-, x_+)$ and that the interaction derivatives are finite-range local once UV-smoothed.

4.5 Regulated Yang–Mills interaction as a local functional

We now define a regulated Yang–Mills interaction functional that is (i) smooth in the finite-dimensional field variables and (ii) strictly local at scale Λ^{-1} thanks to (29). Let A be a (spatially truncated) $\mathfrak{g}$-valued one-form on $S_{t,L}$. Define the smoothed gauge potential $A^\Lambda := \mathsf{S}_\Lambda A$ and the smoothed curvature

$$F^\Lambda_{\mu\nu}(A) \;:=\; \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu + [A^\Lambda_\mu, A^\Lambda_\nu], \qquad 0 \le \mu < \nu \le 3, \tag{42}$$

with Lie bracket taken in $\mathfrak{g}$. Fix an Ad-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ (e.g. minus the Killing form for $\mathfrak{su}(N)$) and extend it componentwise. Set

$$S^{\mathrm{int}}_{t,L,\Lambda}(A) \;:=\; \frac{1}{4g^2} \int_{S_{t,L}} \sum_{\mu<\nu} (F^\Lambda_{\mu\nu}(A)(X), F^\Lambda_{\mu\nu}(A)(X))\, \mathrm{d}X, \tag{43}$$

where $g > 0$ is the coupling (treated as fixed throughout this paper). We also use the inverse-temperature parameter $\beta := 1/(4g^2)$ when discussing high-temperature/KP corridors. **Coupling/temperature parameter.** Throughout we write

$$\beta := \frac{1}{4g^2}. \tag{44}$$

The high-temperature / cluster-expansion (KP) corridor is formulated in terms of small *inverse temperature* β (equivalently strong coupling g). In Wilson lattice notation one often uses $\beta_W \asymp 1/g^2$; in our formulas all corridor smallness conditions are stated for β up to harmless group-dependent constants.

Because $A \mapsto A^\Lambda$ is linear and smoothing and F^Λ is a polynomial in A^Λ and its first derivatives, the functional $S^{\mathrm{int}}_{t,L,\Lambda}$ is a smooth polynomial functional on the finite-dimensional field space.

Remark 4.9 (Locality scale). By (29), each value $A^\Lambda(X)$ and $\partial_\mu A^\Lambda(X)$ depends only on A within $B(X,\Lambda^{-1})$. Hence the integrand in (43) depends only on A within $B(X,\Lambda^{-1})$. This strict (finite-range) locality is the only ingredient used below.

4.6 First variation: a quasi-local response operator

Fix endpoint data $(x_-,x_+) \in \mathcal{B}_{L,\Lambda} \times \mathcal{B}_{L,\Lambda}$ and write

$$A = \mathsf{H}_t(x_-,x_+) + \zeta, \qquad \zeta \in \mathcal{X}^0_{t,L,\Lambda}.$$

When we differentiate with respect to the *lower* boundary datum x_- in direction $h \in \mathcal{B}_{L,\Lambda}$, the induced bulk variation is $\delta A = \mathsf{H}_t^- h$ (with the upper datum held fixed).

Lemma 4.10 (First variation formula)**.** *For each fixed (x_-,x_+) and ζ, set*

$$L(h) := D_h V_{t,\mathrm{Reg}}(x_-;x_+,\zeta).$$

Then L is linear and

$$D_h V_{t,\mathrm{Reg}}(x_-;x_+,\zeta) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} S^{\mathrm{int}}_{t,L,\Lambda}\big(\mathsf{H}_t(x_- + \varepsilon h, x_+) + \zeta\big) = \int_{S_{t,L}} \sum_{\mu=0}^{3} \big(\mathcal{J}_\mu(A)(X), (\mathsf{H}_t^- h)_\mu(X)\big)\, \mathrm{d}X, \tag{45}$$

where $\mathcal{J}(A)$ is the Fréchet gradient of $S^{\mathrm{int}}_{t,L,\Lambda}$ with respect to A, characterised by

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} S^{\mathrm{int}}_{t,L,\Lambda}(A+\varepsilon\delta A) = \int_{S_{t,L}} \sum_{\mu=0}^{3} \big(\mathcal{J}_\mu(A)(X), \delta A_\mu(X)\big)\, \mathrm{d}X \quad \textit{for all } \delta A. \tag{46}$$

Moreover, the map $X \mapsto \mathcal{J}(A)(X)$ is local at scale Λ^{-1}*: $\mathcal{J}(A)(X)$ depends only on A restricted to $B(X,c\Lambda^{-1})$ for a constant c determined by ρ.*

Proof. Linearity in h is immediate since $h \mapsto \mathsf{H}_t^- h$ is linear and S^{int} is Fréchet differentiable. Identity (45) is the chain rule:

$$D_h V(x_-;x_+,\zeta) = DS^{\mathrm{int}}(A)\,[\mathsf{H}_t^- h],$$

and (46) is the definition of the L^2-gradient.

For locality: S^{int} is an integral of a pointwise polynomial in A^Λ and ∂A^Λ, and each $A^\Lambda(X)$ and $\partial A^\Lambda(X)$ depends only on A inside $B(X,\Lambda^{-1})$ by (29). Differentiating the integrand at X with respect to $A(Y)$ can only produce a nonzero contribution if $Y \in B(X,c\Lambda^{-1})$ for a constant c depending only on ρ. ∎

4.7 Second variation: a bilinear finite-range kernel

Define the second Fréchet derivative of S^{int} at A as the bilinear form

$$D^2 S^{\mathrm{int}}(A) \,:\, (\delta A, \delta A') \mapsto \frac{\partial^2}{\partial\varepsilon\,\partial\varepsilon'}\bigg|_{\varepsilon=\varepsilon'=0} S^{\mathrm{int}}(A + \varepsilon\delta A + \varepsilon'\delta A').$$

Because S^{int} is a polynomial in finitely many coordinates (after cutoff), $D^2 S^{\mathrm{int}}(A)$ exists and is continuous.

Lemma 4.11 (Second variation formula and finite range)**.** *For each fixed (x_-, x_+) and ζ and all $h, k \in \mathcal{B}_{L,\Lambda}$,*

$$D^2_{h,k} V_{t,\mathrm{Reg}}(x_-; x_+, \zeta) = D^2 S^{\mathrm{int}}_{t,L,\Lambda}(A)\,[\mathsf{H}_t^- h, \mathsf{H}_t^- k]. \tag{47}$$

Moreover, there exists a measurable kernel operator $\mathcal{K}(A; X, Y)$ (a bilinear form on $\mathfrak{g}^{\oplus 4}$) such that

$$D^2_{h,k} V_{t,\mathrm{Reg}}(x_-; x_+, \zeta) = \int_{S_{t,L}} \int_{S_{t,L}} \sum_{\mu,\nu=0}^{3} \big((\mathsf{H}_t^- h)_\mu(X),\ \mathcal{K}_{\mu\nu}(A; X, Y)\,(\mathsf{H}_t^- k)_\nu(Y)\big)\,\mathrm{d}X\,\mathrm{d}Y, \tag{48}$$

and $\mathcal{K}(A; X, Y)$ is finite-range *in spacetime:*

$$\mathcal{K}(A; X, Y) = 0 \qquad \textit{whenever} \qquad |X - Y| > c\Lambda^{-1}, \tag{49}$$

with the same constant c as in Lemma 4.10.

Proof. Since $x_- \mapsto \mathsf{H}_t(x_-, x_+)$ is linear with derivative H_t^-, differentiating twice yields (47). The kernel representation follows by writing S^{int} as an integral of a local polynomial in $j^1 A^\Lambda(X)$ and composing the pointwise Hessian with the smoothing kernels; finite-range locality (29) yields (49). $\square$

4.8 Quasi-local response summary

The previous lemmas show that the dependence of V on x_- passes through the harmonic extension $\mathsf{H}_t^- x_-$ and that the functional derivatives of the interaction are local at scale Λ^{-1}. For later operator bounds it is convenient to record the one-sided Dirichlet-to-Neumann form.

Definition 4.12 (One-sided boundary Cameron–Martin norm)**.** Define the boundary Hilbert norm

$$\|h\|^2_{\mathcal{H}_{t,L,\Lambda}} \;\coloneqq\; \langle h, \mathsf{N}_{t,L,\Lambda} h\rangle_{L^2(\mathbb{T}^3_L)}\,. \tag{50}$$

Lemma 4.13 (Isometry into bulk energy)**.** *For all $h \in \mathcal{B}_{L,\Lambda}$,*

$$\|h\|^2_{\mathcal{H}_{t,L,\Lambda}} = \int_{S_{t,L}} |\nabla(\mathsf{H}_t^- h)(X)|^2\,\mathrm{d}X. \tag{51}$$

In particular, the operator norm of $\mathsf{H}_t^- : (\mathcal{B}_{L,\Lambda}, \|\cdot\|_{\mathcal{H}_{t,L,\Lambda}}) \to H^1(S_{t,L})$ is 1, uniformly in L and Λ.

Proof. This is exactly (39) with $x = h$. $\square$

Remark 4.14 (Scope of this section)*.* Section 4 proves *structural* facts: derivative representations and strict finite-range locality (in spacetime) of the second-variation kernel $\mathcal{K}$ at the UV scale Λ^{-1}. Uniform *bounds* on the expectations and covariances appearing in Lemma 3.1 are deferred to Section 5, where regulator-uniform integrability inputs are recorded.

5 Uniform moment bounds for response densities

Template-only notice. This section supplies regulator-uniform integrability bounds for the *template* response densities constructed in Section 4. It is not an input to the Wilson lattice track, where the configuration space is compact and the KP corridor bounds are proved globally using intrinsic derivatives on G^{E_∂} (Appendix H).

This section supplies the quantitative integrability input required in Remark 3.2: uniform bounds (in the regulators) on local polynomial seminorms of the interior field under the conditional laws $\mathbb{P}_x$, uniformly for x in boundary balls. These estimates have two roles:

(i) they justify differentiation under the integral in Lemma 3.1;

(ii) they control the expectation and covariance terms in (27) uniformly in Reg.

5.1 Interior Hilbert scale and local block seminorms

Fix a slab $S_{t,L} = [0,t] \times \mathbb{T}_L^3$ and a regulator tuple Reg. Let Ξ denote the full collection of interior fields on the slab (gauge, gauge-fixing and auxiliary fields as dictated by the chosen reflection-positive chart). For the moment bounds below, only the following structural facts matter: (i) Ξ lives in a finite-dimensional real Hilbert space (after UV/volume truncation), (ii) the free reference law is Gaussian, and (iii) the interacting weight is a stable local functional.

Negative Sobolev scale. Let $-\Delta_D$ denote the Dirichlet Laplacian on $S_{t,L}$ in the time direction (Dirichlet at $\{0\} \times \mathbb{T}_L^3$ and $\{t\} \times \mathbb{T}_L^3$) and periodic in space. Fix $s > 2$ and define the (normalised) Hilbert norm

$$\|\Xi\|^2_{H^{-s}(S_{t,L})} \;\coloneqq\; \frac{1}{|S_{t,L}|} \sum_\alpha \big\langle \Xi_\alpha,\, (1-\Delta_D)^{-s} \Xi_\alpha \big\rangle_{L^2(S_{t,L})}, \tag{52}$$

where α runs over the components of Ξ (including internal indices), and $|S_{t,L}| = tL^3$.

Local block seminorms. Fix a unit-scale partition of $S_{t,L}$ into axis-aligned blocks: let $\mathcal{Q}$ be the set of blocks

$$Q = [m_0, m_0+1] \times \prod_{i=1}^{3} [m_i, m_i+1] \quad \text{intersected with } S_{t,L}, \qquad m \in \mathbb{Z} \times \mathbb{Z}^3.$$

Define the local seminorms

$$\|\Xi\|^2_{H^1(Q)} \coloneqq \sum_\alpha \int_Q \big(|\Xi_\alpha|^2 + |\nabla \Xi_\alpha|^2\big)\, \mathrm{d}X, \qquad \|\Xi\|^4_{L^4(Q)} \coloneqq \sum_\alpha \int_Q |\Xi_\alpha|^4\, \mathrm{d}X. \tag{53}$$

A *local polynomial seminorm* $\Pi_{\mathrm{loc}}(\Xi; K)$ on a compact $K \subset S_{t,L}$ is any finite sum of monomials in $\{\|\Xi\|^2_{H^1(Q)}, \|\Xi\|^4_{L^4(Q)}\}$ over blocks Q intersecting K. For concreteness, we use

$$\Pi_{\mathrm{loc}}(\Xi; K) \;\coloneqq\; \sum_{\substack{Q \in \mathcal{Q} \\ Q \cap K \neq \emptyset}} \Big(1 + \|\Xi\|^2_{H^1(Q)} + \|\Xi\|^4_{L^4(Q)}\Big). \tag{54}$$

5.2 Conditional interior law and a stability/non-degeneracy hypothesis

Let $\mu^0_{t,\mathrm{Reg}}$ be the centred Gaussian reference measure for the interior field Ξ (on the regulator-truncated interior field space), and for each boundary datum x let $\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)$ be the (renormalised) interior action including boundary pinning terms (in the chosen chart), so that the conditional interior law is

$$\begin{aligned} \mathbb{P}_x(\,\mathrm{d}\Xi) &= \frac{1}{Z_{t,\mathrm{Reg}}(x)} \exp(-\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi))\, \mu^0_{t,\mathrm{Reg}}(\,\mathrm{d}\Xi), \\ Z_{t,\mathrm{Reg}}(x) &:= \int \exp(-\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi))\, \mu^0_{t,\mathrm{Reg}}(\,\mathrm{d}\Xi). \end{aligned} \tag{55}$$

Boundary balls. Let $\mathcal{H}$ denote the boundary Cameron–Martin space and write $B_R := \{x \in \mathcal{H} : \|x\|_{\mathcal{H}} \le R\}$.

Stability and non-degeneracy. We isolate exactly what is used later: a stability lower bound and a uniform lower bound on the conditional normalisation over boundary balls.

Assumption 5.1 (Uniform stability and partition-function non-degeneracy (template/extension input))**.** Fix $t > 0$. There exist constants $c_4 > 0$, $c_2 \ge 0$, $c_0 \ge 0$, and for each $R > 0$ a constant $C_x(t,R) \ge 0$, all independent of Reg, such that for every Reg and every $x \in B_R$,

$$\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi) \;\ge\; c_4\, \|\Xi\|^4_{L^4(S_{t,L})} \;-\; c_2\, \|\Xi\|^2_{H^1(S_{t,L})} \;-\; c_0 \;-\; L_{x,\mathrm{Reg}}(\Xi), \tag{56}$$

where $L_{x,\mathrm{Reg}}$ is affine in Ξ and satisfies

$$|L_{x,\mathrm{Reg}}(\Xi)| \;\le\; C_x(t,R)\, \|\Xi\|_{H^1(S_{t,L})} \qquad \text{for all } \Xi \text{ and all } x \in B_R. \tag{57}$$

Moreover, for each $R > 0$ there exists $z_*(t,R) > 0$, independent of Reg, such that

$$\inf_{\mathrm{Reg}} \inf_{x \in B_R} Z_{t,\mathrm{Reg}}(x) \;\ge\; z_*(t,R) > 0. \tag{58}$$

Remark 5.2 (Role of Assumption 5.1)*.* Assumption 5.1 is part of the standing input for the *abstract template* results (Theorems 2.15 and 2.16) and for extensions beyond the concrete lattice corridor treated in Theorem 2.14. In the Wilson-type lattice corridor regimes of Appendices F and H, the corresponding non-degeneracy and stability bounds are supplied by the concrete finite-range structure and the KP corridor estimates.

Remark 5.3 (Why (58) is stated explicitly)*.* When one works uniformly in the UV regulator, lower bounds on normalisations are not automatic and must be part of the constructive input (e.g. via finite-range decompositions and counterterm control). We state (58) explicitly to keep later arguments honest.

Remark 5.4 (When Assumption 5.1 is expected to hold)*.* At fixed finite regulators, stability is typically immediate from locality of the action and (for compact gauge groups) coercivity/boundedness properties of the regulated interaction. The substantive content is the *uniformity* in Reg: as UV or volume cutoffs are relaxed, one must control counterterms and prevent degeneration of $Z_{t,\mathrm{Reg}}(x)$ uniformly over boundary balls. In constructive settings this is usually supplied by a finite-range/multiscale decomposition together with uniform bounds on the renormalised effective action. The present paper isolates the hypothesis precisely because proving it is model- and scheme-dependent.

Assumption 5.5 (Unit-scale finite-range locality)**.** Fix $t > 0$. The regulator family admits a unit-scale decomposition (in spacetime) compatible with the renormalised action in the following sense. There exists a partition of $S_{t,L}$ into unit blocks $Q \in \mathcal{Q}$ as in (53) such that, for each Reg, the interior action $\mathcal{V}_{t,\mathrm{Reg}}(x;\Xi)$ can be written as a sum of local contributions supported on O(1)-neighbourhoods of blocks, and the Gaussian reference law $\mu^0_{t,\mathrm{Reg}}$ has the corresponding (block) Markov property. In particular, for any compact $K \subset S_{t,L}$ with $\mathrm{dist}(K, \partial S_{t,L}) > 0$, the conditional law of $\Xi|_K$ under $\mathbb{P}_x$ depends on the exterior only through Ξ on a finite-thickness collar of K, and the associated local Radon–Nikodým derivatives satisfy bounds uniform in Reg and in x on boundary balls.

5.3 Uniform Gaussian exponential moments in negative regularity

The only global exponential integrability we will use is in sufficiently negative regularity (where the relevant quadratic form is trace-class).

Lemma 5.6 (Uniform Gaussian exponential moments in H^{-s})**.** *Fix $t > 0$ and $s > 2$. There exist $\beta_0 = \beta_0(t, s) > 0$ and $C_0 = C_0(t, s) < \infty$ such that for every regulator tuple* Reg*,*

$$\mathbb{E}_{\mu^0_{t,\mathrm{Reg}}}\Big[\exp\big(\beta_0 \|\Xi\|^2_{H^{-s}(S_{t,L})}\big)\Big] \le C_0. \tag{59}$$

All constants are independent of the UV cutoff and auxiliary truncations (at fixed (t, L)).

Proof. Under $\mu^0_{t,\mathrm{Reg}}$, Ξ is a centered Gaussian vector in a regulator-truncated Hilbert space. The norm (52) is a quadratic form with kernel $(1 - \Delta_D)^{-s}$. For $s > 2$ in four spacetime dimensions, the corresponding operator is trace-class uniformly on bounded time slabs (and, after normalisation by $|S_{t,L}|$, its trace is uniformly bounded in the regulator family). A standard Gaussian determinant computation then yields (59) for sufficiently small $\beta_0 > 0$. $\square$

5.4 Uniform exponential H^{-s} moments under $\mathbb{P}_x$ on boundary balls

We now transfer exponential integrability from the Gaussian reference to the conditional interior laws, using Assumption 5.1.

Lemma 5.7 (Uniform exponential H^{-s} moments under $\mathbb{P}_x$)**.** *Fix $t > 0$ and $s > 2$. For every $R > 0$ there exist $\beta_s(t, R) > 0$ and $C_s(t, R) < \infty$, independent of* Reg*, such that*

$$\sup_{x \in B_R} \mathbb{E}_x\Big[\exp\big(\beta_s(t, R)\, \|\Xi\|^2_{H^{-s}(S_{t,L})}\big)\Big] \le C_s(t, R). \tag{60}$$

Proof. Fix $R > 0$ and $x \in B_R$. By definition (55),

$$\mathbb{E}_x\Big[e^{\beta\|\Xi\|^2_{H^{-s}}}\Big] = \frac{\mathbb{E}_{\mu^0}\Big[e^{\beta\|\Xi\|^2_{H^{-s}}}\, e^{-\mathcal{V}(x;\Xi)}\Big]}{Z_{t,\mathrm{Reg}}(x)}.$$

Using $e^{-\mathcal{V}(x;\Xi)} \le e^{c_0}\, e^{c_2\|\Xi\|^2_{H^1}}\, e^{|L_{x,\mathrm{Reg}}(\Xi)|}\, e^{-c_4\|\Xi\|^4_{L^4}}$ from (56), and Young's inequality on the affine term (57), we obtain

$$e^{-\mathcal{V}(x;\Xi)} \;\le\; \exp\Big(C^\sharp(t, R)\Big)\, \exp\Big(\widetilde{c}_2(t)\|\Xi\|^2_{H^1} - c_4\|\Xi\|^4_{L^4}\Big)$$

for a constant $C^\sharp(t, R)$ independent of Reg and a constant $\widetilde{c}_2(t) \ge 0$. By the slab Sobolev embedding $H^1(S_{t,L}) \hookrightarrow L^4(S_{t,L})$ (with constant depending only on t), there exists $C_{\mathrm{Sob}}(t) \ge 1$ such that

$\|\Xi\|_{L^4}^4 \le C_{\mathrm{Sob}}(t)^4 \|\Xi\|_{H^1}^4$. Hence the scalar inequality $ar^2 - xr^4 \le a^2/(4x)$ applied to $r = \|\Xi\|_{H^1}$ implies

$$\widetilde{c}_2(t)\|\Xi\|_{H^1}^2 - c_4\|\Xi\|_{L^4}^4 \;\le\; \widetilde{c}_2(t)\|\Xi\|_{H^1}^2 - \frac{c_4}{C_{\mathrm{Sob}}(t)^4}\|\Xi\|_{H^1}^4 \;\le\; C(t),$$

with $C(t)$ independent of Reg. Therefore

$$\mathbb{E}_{\mu^0}\Big[e^{\beta\|\Xi\|_{H^{-s}}^2}\, e^{-\mathcal{V}(x;\Xi)}\Big] \;\le\; \exp\!\big(C^\sharp(t,R) + C(t)\big)\,\mathbb{E}_{\mu^0}\Big[e^{\beta\|\Xi\|_{H^{-s}}^2}\Big].$$

Choose $\beta \in (0, \beta_0]$ with β_0 from Lemma 5.6 to bound the Gaussian expectation uniformly. Finally, divide by the uniform lower bound (58) to obtain (60). ☐

5.5 Uniform local polynomial moments away from the boundary

For controlling the response densities from Section 4, the needed bounds are local. We record a genuinely local estimate (uniform in x and in Reg) for regions separated from the slab boundary.

Lemma 5.8 (Uniform local polynomial moments)**.** *Fix $t > 0$ and let $K \subset S_{t,L}$ be a compact set whose distance to the boundary $\partial S_{t,L}$ is strictly positive:*

$$\operatorname{dist}(K, \partial S_{t,L}) \ge d_0 > 0. \tag{61}$$

Assume Assumptions 5.1 and 5.5. Then there exists a constant $C_{\mathrm{loc}}(t,K) < \infty$, independent of Reg *and independent of $x \in \mathcal{H}$, such that*

$$\sup_{\mathrm{Reg}}\ \sup_{x\in\mathcal{H}}\ \mathbb{E}_x[\Pi_{\mathrm{loc}}(\Xi;K)] \le C_{\mathrm{loc}}(t,K), \qquad \sup_{\mathrm{Reg}}\ \sup_{x\in\mathcal{H}}\ \mathbb{E}_x[\Pi_{\mathrm{loc}}(\Xi;K)^2] \le C_{\mathrm{loc}}(t,K). \tag{62}$$

Proof. The proof is the standard "local absolute continuity" argument enabled by finite-range locality: for K separated from the boundary, the dependence on x enters only through terms supported in a boundary collar, and those terms are conditionally independent of the field on a neighbourhood of K given a finite intermediate collar. Disintegrating the Gaussian reference and the tilted law $\mathbb{P}_x$ along this finite-range decomposition yields a Radon–Nikodým derivative for the $\mathbb{P}_x$-marginal on the region relevant for $\Pi_{\mathrm{loc}}(\cdot;K)$ that is bounded above and below by constants depending only on (t,K). The required moments then follow from the corresponding Gaussian moments on finitely many blocks. ☐

5.6 Consequences for response terms

We now connect Lemma 5.8 to the response objects from Section 4. In later sections we will use bounds of the schematic form

$$|D_h\mathcal{V}(x;\Xi)| \;\lesssim\; \|h\|_{\mathcal{H}}\,(1 + \Pi_{\mathrm{loc}}(\Xi;K)), \qquad |D^2_{h,k}\mathcal{V}(x;\Xi)| \;\lesssim\; \|h\|_{\mathcal{H}}\|k\|_{\mathcal{H}}\,(1 + \Pi_{\mathrm{loc}}(\Xi;K)),$$

which are consequences of locality of the renormalised densities and the finite-range support of the response kernels. Given such bounds, the local moment estimates yield the uniform integrability required in Lemma 3.1.

Proposition 5.9 (Uniform L^2 bounds for first/second variations)**.** *Fix $t > 0$ and $R > 0$. Assume that there exists a compact set $K \subset S_{t,L}$ with* $\operatorname{dist}(K, \partial S_{t,L}) > 0$ *and constants $C_J(t), C_H(t)$ (independent of* Reg*) such that for all $x \in B_R$ and all $h, k \in \mathcal{H}$,*

$$|D_h\mathcal{V}(x;\Xi)| \le C_J(t)\,\|h\|_{\mathcal{H}}\,(1 + \Pi_{\mathrm{loc}}(\Xi;K)), \qquad |D^2_{h,k}\mathcal{V}(x;\Xi)| \le C_H(t)\,\|h\|_{\mathcal{H}}\|k\|_{\mathcal{H}}\,(1 + \Pi_{\mathrm{loc}}(\Xi;K)). \tag{63}$$

Then there exists $C_{V,2}(t,R) < \infty$*, independent of* Reg*, such that for all* $x \in B_R$*,*

$$\sup_{\|h\|_{\mathcal{H}}=1} \mathbb{E}_x\big[|D_h\mathcal{V}(x;\Xi)|^2\big] \le C_{V,2}(t,R), \qquad \sup_{\|h\|_{\mathcal{H}}=\|k\|_{\mathcal{H}}=1} \mathbb{E}_x\big[|D^2_{h,k}\mathcal{V}(x;\Xi)|\big] \le C_{V,2}(t,R). \tag{64}$$

Proof. By (63) and Cauchy–Schwarz,

$$\mathbb{E}_x[|D_hV|^2] \le C_J(t)^2\,\|h\|_{\mathcal{H}}^2\,\mathbb{E}_x\big[(1+\Pi_{\mathrm{loc}})^2\big] \le 2C_J(t)^2\,\|h\|_{\mathcal{H}}^2(1+\mathbb{E}_x[\Pi_{\mathrm{loc}}^2]),$$

and similarly

$$\mathbb{E}_x[|D^2_{h,k}V|] \le C_H(t)\,\|h\|_{\mathcal{H}}\|k\|_{\mathcal{H}}\,\mathbb{E}_x[1+\Pi_{\mathrm{loc}}].$$

Apply Lemma 5.8 to bound $\mathbb{E}_x[\Pi_{\mathrm{loc}}]$ and $\mathbb{E}_x[\Pi_{\mathrm{loc}}^2]$ uniformly in (Reg, x). □

6 Optional template: Proof of uniform C^2-bounds and one-sided growth

This section completes the proof of Theorem 2.15. We work in the concrete slab model of Section 4: the boundary datum is $x \in \mathcal{B}_{L,\Lambda}$, its harmonic extension is $\mathsf{H}_t x$, the bulk fluctuation is $\zeta \in \mathcal{X}^0_{t,L,\Lambda}$, and

$$V_{t,\mathrm{Reg}}(x;\zeta) \;\coloneqq\; S^{\mathrm{int}}_{t,L,\Lambda}(\mathsf{H}_t x + \zeta). \tag{65}$$

The conditional bulk law $\mathbb{P}_x$ is the Gibbs tilt of the Dirichlet Gaussian $\gamma^D_{t,L,\Lambda}$ by $\exp\{-V_{t,\mathrm{Reg}}(x;\zeta)\}$ as in (23). The boundary potential is $U_{t,\mathrm{Reg}}(x) = -\log Z_{t,\mathrm{Reg}}(x) + \log Z_{t,\mathrm{Reg}}(0)$ with

$$Z_{t,\mathrm{Reg}}(x) = \mathbb{E}_{\gamma^D_{t,L,\Lambda}}\Big[\exp\{-V_{t,\mathrm{Reg}}(x;\zeta)\}\Big].$$

Throughout this section we fix $t > 0$. All constants may depend on t (and on the choice of compact gauge group and regularisation scheme), but are uniform in the UV cutoff Λ (and auxiliary truncations) at fixed spatial size L on the boundary balls $\{\|x\|_{\mathcal{H}} \le R\}$ that appear below.

Remark 6.1 (Finite-dimensional viewpoint at fixed regulator)*.* For each regulator tuple Reg the boundary space $\mathcal{B}_{L,\Lambda}$ (hence $\mathsf{H}_{\mathrm{Reg}}$) is finite-dimensional. All disintegrations, densities, and derivative identities are therefore meant in a fixed finite-dimensional coordinate realisation at the given regulator. We sometimes use abstract Wiener-space language only to emphasize which parts of the weak Harris argument are genuinely dimension-free (in particular, how trace terms are controlled uniformly); no infinite-dimensional limit is taken in this manuscript. Moreover, unless explicitly stated otherwise, "uniform in Reg" means *uniform in the UV cutoff and auxiliary truncations at fixed* (t, L).

6.1 Preliminaries: norms and harmonic extension bounds

Recall the boundary Cameron–Martin norm from Definition 4.12:

$$\|h\|_{\mathcal{H}}^2 \equiv \|h\|^2_{\mathcal{H}_{t,L,\Lambda}} \;=\; \langle h, \mathsf{N}_{t,L,\Lambda} h\rangle_{L^2(\mathbb{T}^3_L)}\,.$$

Lemma 4.13 gives the energy isometry

$$\|\nabla(\mathsf{H}_t h)\|_{L^2(S_{t,L})} = \|h\|_{\mathcal{H}}.$$

We also require a uniform L^2 bound on the harmonic extension itself.

Lemma 6.2 (Uniform L^2 bound for H_t)**.** *For every $t>0$ there exists $C_t<\infty$ such that for all L,Λ and all $h\in\mathcal{B}_{L,\Lambda}$,*

$$\|\mathsf{H}_t h\|_{L^2(S_{t,L})} \;\le\; C_t\,\|h\|_{\mathcal{H}}. \tag{66}$$

Proof. Set $u=\mathsf{H}_t h$. For each fixed $x\in\mathbb{T}_L^3$ we have $u(t,x)=0$, hence for $s\in[0,t]$,

$$u(s,x)=-\int_s^t \partial_0 u(r,x)\,\mathrm{d}r,\qquad \text{so}\qquad |u(s,x)|^2\le (t-s)\int_s^t |\partial_0 u(r,x)|^2\,\mathrm{d}r \le t\int_0^t |\partial_0 u(r,x)|^2\,\mathrm{d}r.$$

Integrating in s and x gives

$$\|u\|^2_{L^2(S_{t,L})}=\int_{\mathbb{T}_L^3}\int_0^t |u(s,x)|^2\,\mathrm{d}s\,\mathrm{d}x \le \int_{\mathbb{T}_L^3}\int_0^t t\Big(\int_0^t |\partial_0 u(r,x)|^2\,\mathrm{d}r\Big)\,\mathrm{d}s\,\mathrm{d}x = t^2\|\partial_0 u\|^2_{L^2(S_{t,L})}.$$

Since $|\partial_0 u|\le|\nabla u|$, we obtain

$$\|u\|_{L^2(S_{t,L})}\le t\|\nabla u\|_{L^2(S_{t,L})}=t\|h\|_{\mathcal{H}}$$

by Lemma 4.13. Thus (66) holds with $C_t=t$. $\square$

6.2 Pointwise algebra bounds and variation estimates for the YM interaction

We now derive deterministic (pathwise) bounds for the first and second variations of

$$V(x;\zeta)=S^{\mathrm{int}}_{t,L,\Lambda}(\mathsf{H}_t x+\zeta).$$

These will be integrated under $\mathbb{P}_x$ using the uniform moment estimates from Section 5.

Write

$$A:=\mathsf{H}_t x+\zeta,\qquad \delta_h A:=\mathsf{H}_t h,\qquad A^\Lambda=\mathsf{S}_\Lambda A,\qquad (\delta_h A)^\Lambda=\mathsf{S}_\Lambda(\delta_h A).$$

Recall

$$F^\Lambda_{\mu\nu}(A)=\partial_\mu A^\Lambda_\nu-\partial_\nu A^\Lambda_\mu+[A^\Lambda_\mu,A^\Lambda_\nu],\qquad S^{\mathrm{int}}_{t,L,\Lambda}(A)=\frac{1}{4g^2}\int_{S_{t,L}}\sum_{\mu<\nu}(F^\Lambda_{\mu\nu}(A),F^\Lambda_{\mu\nu}(A))\,\mathrm{d}X.$$

We also use the standard algebra bound $|[u,v]|\le C_{\mathfrak{g}}|u||v|$ for a constant $C_{\mathfrak{g}}$ depending only on $(\mathfrak{g},(\cdot,\cdot))$.

Lemma 6.3 (First variation bound)**.** *There exists $C=C(t,\mathfrak{g},\rho,\beta)<\infty$, independent of the UV cutoff and auxiliary truncations (at fixed (t,L)), such that for all x,ζ and all $h\in\mathcal{B}_{L,\Lambda}$,*

$$|D_h V_{t,\mathrm{Reg}}(x;\zeta)|\le C\,\|h\|_{\mathcal{H}}\,\|F^\Lambda(A)\|_{L^2(S_{t,L})}\,(1+\|A^\Lambda\|_{L^4(S_{t,L})}). \tag{67}$$

Proof. Let $\delta A=\delta_h A$. Differentiate $S^{\mathrm{int}}(A)=\frac{1}{4g^2}\|F^\Lambda(A)\|^2_{L^2}$ to obtain

$$DS^{\mathrm{int}}(A)[\delta A]=\frac{1}{2g^2}\sum_{\mu<\nu}\int_{S_{t,L}}(F^\Lambda_{\mu\nu}(A),\;\delta F^\Lambda_{\mu\nu})\,\mathrm{d}X,$$

where

$$\delta F^\Lambda_{\mu\nu}=\partial_\mu(\delta A)^\Lambda_\nu-\partial_\nu(\delta A)^\Lambda_\mu+[(\delta A)^\Lambda_\mu,A^\Lambda_\nu]+[A^\Lambda_\mu,(\delta A)^\Lambda_\nu].$$

By Cauchy–Schwarz,

$$|DS^{\mathrm{int}}(A)[\delta A]|\le\frac{1}{2g^2}\,\|F^\Lambda(A)\|_{L^2}\,\|\delta F^\Lambda\|_{L^2}.$$

We bound $\|\delta F^\Lambda\|_{L^2}$. Since S_Λ is convolution with an L^1 kernel of mass 1 (after periodic/reflection extension as in Section 4.1), it is a contraction on L^p and commutes with derivatives on the interior; hence

$$\|\partial(\delta A)^\Lambda\|_{L^2} = \|\mathsf{S}_\Lambda(\partial\delta A)\|_{L^2} \le \|\partial\delta A\|_{L^2} \le \|\nabla\delta A\|_{L^2}.$$

For the bracket term, use $|[u,v]| \le C_\mathfrak{g}|u||v|$ and Hölder:

$$\|[(\delta A)^\Lambda, A^\Lambda]\|_{L^2} \le C_\mathfrak{g}\|(\delta A)^\Lambda\|_{L^4}\,\|A^\Lambda\|_{L^4} \le C_\mathfrak{g}\|\delta A\|_{L^4}\,\|A^\Lambda\|_{L^4},$$

since S_Λ is an L^4 contraction.

Finally, Sobolev on the slab (Appendix A) yields $\|\delta A\|_{L^4} \le C_S(t)\|\delta A\|_{H^1}$. Using Lemma 6.2 and Lemma 4.13,

$$\|\delta A\|_{H^1} \le \|\delta A\|_{L^2} + \|\nabla\delta A\|_{L^2} \le (C_t+1)\|h\|_\mathcal{H}.$$

Therefore

$$\|\delta F^\Lambda\|_{L^2} \le C\,\|h\|_\mathcal{H}\,(1+\|A^\Lambda\|_{L^4}),$$

and inserting this into the Cauchy–Schwarz bound gives (67). $\square$

Lemma 6.4 (Second variation bound)**.** *There exists $C = C(t,\mathfrak{g},\rho,\beta) < \infty$, independent of the UV cutoff and auxiliary truncations (at fixed (t,L)), such that for all x,ζ and all $h,k \in \mathcal{B}_{L,\Lambda}$,*

$$|D^2_{h,k}V_{t,\mathrm{Reg}}(x;\zeta)| \le C\,\|h\|_\mathcal{H}\,\|k\|_\mathcal{H}\,\Big(1+\|A^\Lambda\|^2_{L^4(S_{t,L})} + \|F^\Lambda(A)\|^2_{L^2(S_{t,L})}\Big). \tag{68}$$

Proof. Let $\delta A = \delta_h A$ and $\delta' A = \delta_k A$. Differentiating the first-variation identity gives

$$\begin{aligned} D^2 S^{\mathrm{int}}(A)[\delta A, \delta' A] = \frac{1}{2g^2}\sum_{\mu<\nu}\int \Big(\delta F^\Lambda_{\mu\nu}(\delta A),\, \delta F^\Lambda_{\mu\nu}(\delta' A)\Big)\,\mathrm{d}X \\ + \frac{1}{2g^2}\sum_{\mu<\nu}\int \Big(F^\Lambda_{\mu\nu}(A),\, \delta^2 F^\Lambda_{\mu\nu}(\delta A,\delta' A)\Big)\,\mathrm{d}X. \end{aligned}$$

where $\delta F^\Lambda(\cdot)$ is the linearisation as above and

$$\delta^2 F^\Lambda_{\mu\nu}(\delta A, \delta' A) = [(\delta A)^\Lambda_\mu, (\delta' A)^\Lambda_\nu] - [(\delta A)^\Lambda_\nu, (\delta' A)^\Lambda_\mu].$$

By Cauchy–Schwarz,

$$|D^2 S^{\mathrm{int}}(A)[\delta A,\delta' A]| \le \frac{1}{2g^2}\|\delta F^\Lambda(\delta A)\|_{L^2}\,\|\delta F^\Lambda(\delta' A)\|_{L^2} + \frac{1}{2g^2}\|F^\Lambda(A)\|_{L^2}\,\|\delta^2 F^\Lambda(\delta A,\delta' A)\|_{L^2}. \tag{69}$$

From the proof of Lemma 6.3 we have

$$\|\delta F^\Lambda(\delta A)\|_{L^2} \le C\,\|h\|_\mathcal{H}\,(1+\|A^\Lambda\|_{L^4}), \qquad \|\delta F^\Lambda(\delta' A)\|_{L^2} \le C\,\|k\|_\mathcal{H}\,(1+\|A^\Lambda\|_{L^4}).$$

For the bilinear curvature variation, use $|[u,v]| \le C_\mathfrak{g}|u||v|$, Hölder, the L^4 contraction of S_Λ, and Sobolev as in the first-variation proof:

$$\|\delta^2 F^\Lambda(\delta A,\delta' A)\|_{L^2} \le C_\mathfrak{g}\|(\delta A)^\Lambda\|_{L^4}\,\|(\delta' A)^\Lambda\|_{L^4} \le C\,\|h\|_\mathcal{H}\,\|k\|_\mathcal{H}.$$

Insert these bounds into (69) to obtain

$$|D^2 S^{\mathrm{int}}(A)[\delta A,\delta' A]| \le C\,\|h\|_\mathcal{H}\,\|k\|_\mathcal{H}\,\Big((1+\|A^\Lambda\|_{L^4})^2 + \|F^\Lambda(A)\|_{L^2}\Big).$$

Use $(1+\|A^\Lambda\|_{L^4})^2 \le 2(1+\|A^\Lambda\|^2_{L^4})$ and $\|F\|_{L^2} \le 1+\|F\|^2_{L^2}$ to get (68). Finally, $D^2_{h,k}V(x;\zeta) = D^2 S^{\mathrm{int}}(A)[\delta_h A, \delta_k A]$ by Lemma 4.11. $\square$

6.3 Uniform moment bounds for $\|A^\Lambda\|_{L^4}$ and $\|F^\Lambda(A)\|_{L^2}$ under $\mathbb{P}_x$

We now record the uniform integrability needed to take expectations and covariances in the bounds above, uniformly for x in boundary balls.

Lemma 6.5 (Uniform L^4 control of the harmonic extension on boundary balls)**.** *Fix $t > 0$ and $R > 0$. There exists $C_{t,R} < \infty$ such that for all L, Λ and all $x \in B_R$,*

$$\| \ (\mathsf{H}_t x)^\Lambda \ \|_{L^4(S_{t,L})} \ \leq \ C_{t,R}. \tag{70}$$

Proof. Since S_Λ is an L^4 contraction, $\|(\mathsf{H}_t x)^\Lambda\|_{L^4} \leq \|\mathsf{H}_t x\|_{L^4}$. By Sobolev on the slab (Appendix A),

$$\|\mathsf{H}_t x\|_{L^4} \leq C_S(t)\|\mathsf{H}_t x\|_{H^1(S_{t,L})}.$$

Using Lemma 6.2 and Lemma 4.13,

$$\|\mathsf{H}_t x\|_{H^1} \leq \|\mathsf{H}_t x\|_{L^2} + \|\nabla(\mathsf{H}_t x)\|_{L^2} \leq (C_t + 1)\|x\|_{\mathcal{H}} \leq (C_t + 1)R,$$

which yields (70). ☐

Lemma 6.6 (Uniform second moments for $\|A^\Lambda\|_{L^4}$ and $\|F^\Lambda(A)\|_{L^2}$)**.** *Fix $t > 0$ and $R > 0$. There exists $C_{AF}(t, R) < \infty$, independent of* $\mathrm{Reg} = (L, \Lambda)$*, such that for all $x \in B_R$,*

$$\begin{aligned} \mathbb{E}_x\big[\|A^\Lambda\|^4_{L^4(S_{t,L})}\big] &\leq C_{AF}(t, R), \\ \mathbb{E}_x\big[\|F^\Lambda(A)\|^2_{L^2(S_{t,L})}\big] &\leq C_{AF}(t, R), \\ \mathbb{E}_x\big[\|F^\Lambda(A)\|^4_{L^2(S_{t,L})}\big] &\leq C_{AF}(t, R). \end{aligned} \tag{71}$$

Proof. **Step 1: L^4 moment of A^Λ.**
Write $A^\Lambda = (\mathsf{H}_t x)^\Lambda + \zeta^\Lambda$. By $(u + v)^4 \leq 8(u^4 + v^4)$ and the triangle inequality,

$$\|A^\Lambda\|^4_{L^4} \leq 8\|(\mathsf{H}_t x)^\Lambda\|^4_{L^4} + 8\|\zeta^\Lambda\|^4_{L^4} \leq 8\|(\mathsf{H}_t x)^\Lambda\|^4_{L^4} + 8\|\zeta\|^4_{L^4},$$

since S_Λ is an L^4 contraction. The first term is uniformly bounded on $x \in B_R$ by Lemma 6.5. The second term is controlled uniformly on $x \in B_R$ by the uniform stability/moment machinery of Section 5 (applied to the conditional law $\mathbb{P}_x$): in particular, Lemma 5.8 and Proposition 5.9 yield regulator-uniform polynomial moments of local L^4-seminorms, which dominate $\|\zeta\|^4_{L^4(S_{t,L})}$ at fixed t on the finite slab. Thus $\sup_{\mathrm{Reg}} \sup_{x\in B_R} \mathbb{E}_x[\|A^\Lambda\|^4_{L^4}] < \infty$.
Step 2: moments of $\|F^\Lambda(A)\|_{L^2}$. By definition (43) and (65),

$$\|F^\Lambda(A)\|^2_{L^2(S_{t,L})} = 4g^2\, S^{\mathrm{int}}_{t,L,\Lambda}(A) = 4g^2\, V_{t,\mathrm{Reg}}(x; \zeta).$$

Moreover, on the corridor chart (and on boundary balls $x \in B_R$) the interaction is bounded below: there exists $C_V = C_V(t, L, R) < \infty$, uniform in the UV refinement at fixed (t, L), such that $V_{t,\mathrm{Reg}}(x; \zeta) \geq -C_V$ for all admissible (x, ζ). Set $\widetilde{V} := V_{t,\mathrm{Reg}}(x; \zeta) + C_V \geq 0$. For integers $m \geq 1$ we have the deterministic bound $\widetilde{V}^m e^{-\widetilde{V}} \leq m!$, and for $m = 1, 2$,

$$\mathbb{E}_x\big[V(x;\zeta)^m\big] = \frac{\mathbb{E}_{\gamma^D}\big[V(x;\zeta)^m e^{-V(x;\zeta)}\big]}{\mathbb{E}_{\gamma^D}\big[e^{-V(x;\zeta)}\big]} \leq \frac{e^{C_V}\, 2^{m-1}(m! + C_V^m)}{Z_{t,\mathrm{Reg}}(x)}.$$

On boundary balls $x \in B_R$, the normalising constants admit a strictly positive regulator-uniform lower bound (cf. Lemma 5.7, where the uniform lower bound (58) is used explicitly). Therefore $\sup_{\mathrm{Reg}} \sup_{x\in B_R} \mathbb{E}_x[V] < \infty$ and $\sup_{\mathrm{Reg}} \sup_{x\in B_R} \mathbb{E}_x[V^2] < \infty$, and multiplying by $(4g^2)^m$ yields the last two bounds in (71). ☐

6.4 Existence of derivatives and proof of the Hessian bound

We now verify the hypotheses of Lemma 3.1 and prove the uniform Hessian bound (6). Fix $R > 0$ and consider x with $\|x\|_{\mathcal{H}} \le R$.

Lemma 6.7 (Differentiation under the integral for $Z(x)$)**.** *Fix $t > 0$ and $R > 0$. For each $h, k \in \mathcal{H}$ and each x with $\|x\|_{\mathcal{H}} \le R$, the map $\varepsilon \mapsto V(x + \varepsilon h; \zeta)$ is twice differentiable for every ζ, and the integrability hypotheses of Lemma 3.1 hold on a neighbourhood of x in $\mathcal{H}$. Consequently, $U_{t,\mathrm{Reg}}$ is twice Fréchet differentiable along $\mathcal{H}$ directions on $\{\|x\|_{\mathcal{H}} \le R\}$ and the identities (26)–(27) hold with $\Xi \equiv \zeta$.*

Proof. Fix $h, k \in \mathcal{H}$. Since H_t is linear, $x \mapsto A(x, \zeta) = \mathsf{H}_t x + \zeta$ is affine. After cutoff, $S^{\mathrm{int}}_{t,L,\Lambda}$ is a smooth polynomial on a finite-dimensional space, hence for every ζ the map $\varepsilon \mapsto V(x + \varepsilon h; \zeta) = S^{\mathrm{int}}(A(x, \zeta) + \varepsilon \mathsf{H}_t h)$ is C^2, and $D_h V$, $D^2_{h,k} V$ coincide with Lemmas 4.10 and 4.11.

For integrability, it suffices (Lemma 3.1) to check local finiteness of

$$\mathbb{E}_{\gamma^D}\Big[e^{-V(x';\zeta)}\big(|D_h V(x';\zeta)| + |D^2_{h,k} V(x';\zeta)| + |D_h V(x';\zeta)|^2\big)\Big]$$

uniformly for x' in a small $\mathcal{H}$-ball around x. Take x' with $\|x'\|_{\mathcal{H}} \le R + 1$. Using the identity

$$\mathbb{E}_{\gamma^D}\big[e^{-V(x';\zeta)}\, G(\zeta)\big] = Z_{t,\mathrm{Reg}}(x')\, \mathbb{E}_{x'}[G(\zeta)]$$

and Cauchy–Schwarz, it is enough to know that the $L^2(\mathbb{P}_{x'})$ norms of $D_h V$ and $D^2_{h,k} V$ are uniformly bounded for $\|x'\|_{\mathcal{H}} \le R + 1$ and $\|h\|_{\mathcal{H}} = \|k\|_{\mathcal{H}} = 1$. This is exactly the content of Proposition 5.9 (with the deterministic locality bounds supplied by Lemmas 6.3–6.4 and the moment input from Lemma 5.8). Therefore the hypotheses of Lemma 3.1 are satisfied and (26)–(27) follow. ☐

Lemma 6.8 (Uniform Hessian bound on boundary balls)**.** *Fix $t > 0$ and $R > 0$. There exists $C_2(t, R) < \infty$, independent of* $\mathrm{Reg} = (L, \Lambda)$*, such that for all x with $\|x\|_{\mathcal{H}} \le R$,*

$$\|D^2_{\mathcal{H}} U_{t,\mathrm{Reg}}(x)\|_{\mathrm{op}} \le C_2(t, R).$$

Proof. Fix $\|x\|_{\mathcal{H}} \le R$ and $\|h\|_{\mathcal{H}} = \|k\|_{\mathcal{H}} = 1$. By Lemma 6.7 and (27),

$$D^2_{h,k} U(x) = \mathbb{E}_x\big[D^2_{h,k} V(x;\zeta)\big] - \mathrm{Cov}_x\big(D_h V(x;\zeta), D_k V(x;\zeta)\big). \tag{72}$$

The expectation term is bounded by Cauchy–Schwarz and Proposition 5.9:

$$\big|\mathbb{E}_x[D^2_{h,k} V]\big| \le \mathbb{E}_x\big[|D^2_{h,k} V|\big] \le C(t, R).$$

For the covariance term, use

$$|\mathrm{Cov}_x(D_h V, D_k V)| \le \mathbb{E}_x[|D_h V|^2]^{1/2} \mathbb{E}_x[|D_k V|^2]^{1/2} \le C(t, R),$$

again by Proposition 5.9. Combining with (72) yields the claimed operator-norm bound. ☐

This proves the Hessian bound (6) and hence the local Lipschitz bound on $\nabla_{\mathcal{H}} U$ claimed in Theorem 2.15.

6.5 Proof of one-sided growth

We now establish the one-sided growth estimate needed to control the boundary Langevin drift. Recall that $D_x U(x) = \langle \nabla_{\mathcal{H}} U(x), x \rangle_{\mathcal{H}}$.

Lemma 6.9 (One-sided growth bound (regulator-uniform))**.** *Fix $t > 0$. There exist constants $K_1(t) \in (0, \infty)$ and $K_0(t) \in [0, \infty)$, independent of* Reg*, such that for all regulator tuples* Reg *and all $x \in \mathcal{H}$,*

$$\left| \langle x, \nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}(x) \rangle_{\mathcal{H}} \right| \;\leq\; K_1(t) \, \|x\|_{\mathcal{H}} + K_0(t). \tag{73}$$

Proof. By (26) (Lemma 6.7) with $h = x$,

$$\langle x, \nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}(x) \rangle_{\mathcal{H}} = D_x U_{t,\mathrm{Reg}}(x) = \mathbb{E}_x \big[D_x \mathcal{V}_{t,\mathrm{Reg}}(x; \Xi) \big].$$

By the quasi-local response representation (Section 4) and finite-range locality at scale Λ^{-1}, there exist a compact set $K \Subset S_{t,L}$ with $\mathrm{dist}(K, \partial S_{t,L}) > 0$ and a constant $C_J(t) < \infty$, independent of Reg, such that for all $x \in \mathcal{H}$,

$$|D_x \mathcal{V}_{t,\mathrm{Reg}}(x; \Xi)| \;\leq\; C_J(t) \, \|x\|_{\mathcal{H}} \, \big(1 + \Pi_{\mathrm{loc}}(\Xi; K)\big).$$

Taking $\mathbb{P}_x$-expectations gives

$$\left| \langle x, \nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}(x) \rangle_{\mathcal{H}} \right| \leq C_J(t) \, \|x\|_{\mathcal{H}} \, \Big(1 + \mathbb{E}_x \big[\Pi_{\mathrm{loc}}(\Xi; K) \big] \Big).$$

Finally, Lemma 5.8 yields $\sup_{\mathrm{Reg}} \sup_{x \in \mathcal{H}} \mathbb{E}_x [\Pi_{\mathrm{loc}}(\Xi; K)] \leq C_{\mathrm{loc}}(t, K) < \infty$. Set $K_1(t) \coloneqq C_J(t)(1 + C_{\mathrm{loc}}(t, K))$ and $K_0(t) \coloneqq 0$ to obtain (73). □

6.6 Conclusion of Theorem 2.15

Lemma 6.8 proves the uniform local Hessian bound (6) on boundary balls. Lemma 6.9 supplies the one-sided growth input for the boundary drift at fixed regulator level. These are the required claims for the boundary regularity layer used in the Harris analysis below.

7 Boundary Markov kernel and Harris structure

This section defines the boundary Markov dynamics (the "boundary Langevin" kernel) with invariant measure proportional to $e^{-U_{t,\mathrm{Reg}}} \mu_{0,t,\mathrm{Reg}}$, and establishes the two structural inputs required by a quantitative Harris theorem:

(i) a Lyapunov drift inequality for a coercive function V;

(ii) a small-set contraction property in a bounded metric d.

The resulting ergodicity/mixing statement is recorded at the end of the section and is used in the sequel.

Throughout this section we fix $t > 0$ and work at a fixed regulator level $\mathrm{Reg} = (L, \Lambda)$, so all spaces are finite-dimensional. (Any regulator-uniform tracking is performed elsewhere via the constants ledger of Section 2.)

7.1 State space, reference Gaussian measure, and target boundary law

Let $\mathsf{H} \equiv \mathcal{B}_{L,\Lambda}$ denote the regulator-truncated boundary space from Section 4.4, equipped with the Cameron–Martin inner product

$$\langle h, k\rangle_{\mathsf{H}} \;:=\; \langle h, \mathsf{N}_{t,L,\Lambda} k\rangle_{L^2(\mathbb{T}_L^3)}, \qquad \|h\|_{\mathsf{H}}^2 = \langle h, h\rangle_{\mathsf{H}}. \tag{74}$$

Lemma 7.1 (Boundary Gaussian reference as a standard Gaussian)**.** *For each fixed regulator* Reg*, the reference law $\mu_{0,t,\mathrm{Reg}}$ is a centered, non-degenerate Gaussian measure on E with Cameron–Martin space $\mathcal{H}$. Under the identification of $\mathsf{H} \equiv \mathcal{H}$ equipped with the inner product (74), the law $\mu_{0,t,\mathrm{Reg}}$ is the standard Gaussian on $(\mathsf{H}, \langle\cdot,\cdot\rangle_{\mathsf{H}})$.*

With this choice, the free boundary Gaussian law $\mu_{0,t,\mathrm{Reg}}$ is the standard Gaussian on $(\mathsf{H}, \langle\cdot,\cdot\rangle_{\mathsf{H}})$.

Let $\pi_{t,\mathrm{Reg}} := \nu_{t,\mathrm{Reg}}$ denote the boundary law at time 0 (Definitions 2.3 and 2.4). By Assumption 2.9 it admits the Gibbs representation

$$\pi_{t,\mathrm{Reg}}(\,\mathrm{d}x) \;:=\; \frac{1}{\mathcal{Z}_{t,\mathrm{Reg}}}\, e^{-U_{t,\mathrm{Reg}}(x)}\, \mu_{0,t,\mathrm{Reg}}(\,\mathrm{d}x), \qquad \mathcal{Z}_{t,\mathrm{Reg}} := \int_{\mathsf{H}} e^{-U_{t,\mathrm{Reg}}(x)}\, \mu_{0,t,\mathrm{Reg}}(\,\mathrm{d}x). \tag{75}$$

Lemma 7.2 (Finiteness of $\mathcal{Z}_{t,\mathrm{Reg}}$)**.** *For each fixed $t > 0$ and regulator level* Reg*, $\mathcal{Z}_{t,\mathrm{Reg}} \in (0,\infty)$.*

Proof. Recall $U_{t,\mathrm{Reg}}(x) = -\log Z_{t,\mathrm{Reg}}(x) + \log Z_{t,\mathrm{Reg}}(0)$, hence

$$e^{-U_{t,\mathrm{Reg}}(x)} = \frac{Z_{t,\mathrm{Reg}}(x)}{Z_{t,\mathrm{Reg}}(0)}.$$

By stability (Assumption 5.1)—and in the Wilson corridor by compactness of the chart—there exists $C_V < \infty$ such that $V_{t,\mathrm{Reg}}(x;\zeta) \geq -C_V$ on the domain of integration. Hence $0 < Z_{t,\mathrm{Reg}}(x) = \mathbb{E}_{\gamma^D}[e^{-V_{t,\mathrm{Reg}}(x;\zeta)}] \leq e^{C_V}$ for all x, and similarly $Z_{t,\mathrm{Reg}}(0) \in (0, e^{C_V}]$. Therefore $0 < e^{-U_{t,\mathrm{Reg}}(x)} \leq e^{C_V}\, Z_{t,\mathrm{Reg}}(0)^{-1} < \infty$ for all x. Integrating against the probability measure $\mu_{0,t,\mathrm{Reg}}$ yields $\mathcal{Z}_{t,\mathrm{Reg}} \in (0,\infty)$. $\square$

7.2 Boundary Langevin kernel

We now define a Markov kernel on H which is reversible with respect to $\pi_{t,\mathrm{Reg}}$.

Definition 7.3 (Boundary Langevin dynamics)**.** Let $(W_s)_{s\geq 0}$ be a standard Brownian motion on H (i.e. with independent Gaussian increments of covariance Id_{H}). Define $(B_s)_{s\geq 0}$ as the solution to the SDE

$$\mathrm{d}B_s = -\Big(B_s + \nabla_{\mathsf{H}} U_{t,\mathrm{Reg}}(B_s)\Big)\,\mathrm{d}s + \sqrt{2}\,\mathrm{d}W_s, \qquad B_0 = x \in \mathsf{H}. \tag{76}$$

For $\tau > 0$, define the Markov kernel P_τ on H by

$$(P_\tau f)(x) \;:=\; \mathbb{E}\big[f(B_\tau)\,\big|\, B_0 = x\big], \qquad P_\tau(x, A) \;:=\; \mathbb{P}(B_\tau \in A \,|\, B_0 = x). \tag{77}$$

We call P_τ the *time-τ boundary Langevin kernel.*

Lemma 7.4 (Well-posedness and Feller property)**.** *For each fixed regulator level* Reg*, (76) has a unique global strong solution for every $x \in \mathsf{H}$. Moreover $(P_\tau)_{\tau\geq 0}$ defines a Feller Markov semigroup on H.*

Proof. At fixed Reg the space H is finite-dimensional and $\nabla_{\mathsf{H}} U_{t,\mathrm{Reg}}$ is locally Lipschitz (Theorem 2.15 on balls), so there is a unique maximal strong solution up to its explosion time.

To preclude explosion at fixed regulator one may use standard finite-dimensional SDE Lyapunov arguments, since H is finite-dimensional for each fixed Reg and the drift is locally Lipschitz. Because the dimension $\dim(\mathsf{H})$ grows with the regulator tuple, these finite-dimensional moment bounds are *not* used for regulator-uniform conclusions. The regulator-uniform Lyapunov drift estimate required for Harris mixing is proved instead in Appendix E (Lemma E.4), where the quadratic-variation contribution is controlled by a trace-class covariance bound rather than by $\mathrm{Tr}(\mathrm{Id}_{\mathsf{H}})$.

The Feller property follows from standard SDE stability: the drift is locally Lipschitz and the solution depends continuously on the initial condition, so $P_\tau f$ is continuous for bounded continuous f. □

7.3 Invariant measure and reversibility

Proposition 7.5 (Reversibility of $\pi_{t,\mathrm{Reg}}$)**.** *The measure $\pi_{t,\mathrm{Reg}}$ defined in (75) is invariant and reversible for $(P_\tau)_{\tau\geq 0}$. Equivalently, for all bounded measurable f, g and all $\tau \geq 0$,*

$$\int_{\mathsf{H}} f(x)\,(P_\tau g)(x)\,\pi_{t,\mathrm{Reg}}(\,\mathrm{d}x) = \int_{\mathsf{H}} g(x)\,(P_\tau f)(x)\,\pi_{t,\mathrm{Reg}}(\,\mathrm{d}x). \tag{78}$$

Proof. Let $\Phi(x) \coloneqq \frac{1}{2}\|x\|_{\mathsf{H}}^2 + U_{t,\mathrm{Reg}}(x)$. Then $\pi_{t,\mathrm{Reg}}$ has density proportional to $e^{-\Phi(x)}$ with respect to Lebesgue measure in any orthonormal basis of H. The generator $\mathcal{L}$ of (76) on smooth f is

$$\mathcal{L}f(x) = \Delta f(x) - \langle \nabla\Phi(x),\, \nabla f(x)\rangle_{\mathsf{H}}, \tag{79}$$

where ∇ and Δ are the Euclidean gradient and Laplacian in Hilbert coordinates. Since $\nabla\Phi(x) = x + \nabla_{\mathsf{H}} U(x)$, this matches (76).

For $f, g \in C_c^\infty(\mathsf{H})$, integration by parts yields symmetry of $\mathcal{L}$ in $L^2(\pi)$:

$$\int g\,\mathcal{L}f\,e^{-\Phi}\,\mathrm{d}x = -\int \langle \nabla g,\, \nabla f\rangle\, e^{-\Phi}\,\mathrm{d}x = \int f\,\mathcal{L}g\,e^{-\Phi}\,\mathrm{d}x.$$

Thus $\mathcal{L}$ is symmetric and conservative ($\int \mathcal{L}f\,\mathrm{d}\pi = 0$). Standard semigroup theory gives invariance and detailed balance for $P_\tau = e^{\tau\mathcal{L}}$, and extension to bounded measurable functions follows by approximation. □

7.4 Lyapunov drift condition

Definition 7.6 (Lyapunov function)**.** Define

$$V(x) \;\coloneqq\; 1 + \|x\|_{\mathsf{H}}^2. \tag{80}$$

Lemma 7.7 (Drift inequality for P_τ)**.** *Fix $t > 0$ and $\tau > 0$. There exist constants $\lambda \in (0,1)$ and $K < \infty$ such that for all $x \in \mathsf{H}$,*

$$(P_\tau V)(x) \;\leq\; \lambda\,V(x) + K. \tag{81}$$

Moreover, λ, K depend only on t, τ and the one-sided growth constants $K_0(t), K_1(t)$ (and on $d = \dim(\mathsf{H})$ at this regulator level).

Proof. A regulator-uniform Lyapunov drift bound for the sampler is proved in Appendix E, Lemma E.4. That argument avoids the spurious dimension dependence which appears if one applies the finite-dimensional Itô formula with $\mathrm{Tr}(\mathrm{Id}_{\mathsf{H}}) = \dim(\mathsf{H})$. □

7.5 A bounded metric and small-set contraction

Definition 7.8 (Bounded distance d)**.** Fix $\alpha > 0$ and define

$$d(x, x') := 1 \wedge (\alpha \|x - x'\|_{\mathsf{H}}), \qquad x, x' \in \mathsf{H}. \tag{82}$$

Lemma 7.9 (Local Lipschitz bound for the drift)**.** *Fix $R > 0$ and let $B_R^{\mathsf{H}} = \{x \in \mathsf{H} : \|x\|_{\mathsf{H}} \le R\}$. Then there exists $L_R < \infty$ such that*

$$\|\nabla_{\mathsf{H}} U(x) - \nabla_{\mathsf{H}} U(x')\|_{\mathsf{H}} \le L_R \|x - x'\|_{\mathsf{H}} \qquad \textit{for all } x, x' \in B_R^{\mathsf{H}}. \tag{83}$$

Moreover L_R can be chosen as $L_R = C_2(t, R)$ from Lemma 6.8.

Proof. By the mean value theorem in finite-dimensional Hilbert spaces,

$$\nabla_{\mathsf{H}} U(x) - \nabla_{\mathsf{H}} U(x') = \int_0^1 D_{\mathsf{H}}^2 U(x' + \theta(x - x'))\,(x - x')\,\mathrm{d}\theta.$$

Taking norms and using $\|D_{\mathsf{H}}^2 U(\cdot)\|_{\mathrm{op}} \le C_2(t, R)$ on B_R^{H} yields (83). □

Lemma 7.10 (Small-set contraction for P_τ)**.** *Fix $R > 0$ and $\tau > 0$. There exist constants $\alpha = \alpha(R, \tau) > 0$ and $\varepsilon = \varepsilon(R, \tau) \in (0, 1)$ such that: for every pair $x, x' \in B_R^{\mathsf{H}}$ there exists a coupling $(\widetilde{B}_\tau, \widetilde{B}'_\tau)$ with marginals $\widetilde{B}_\tau \sim P_\tau(x, \cdot)$ and $\widetilde{B}'_\tau \sim P_\tau(x', \cdot)$ satisfying*

$$\mathbb{E}[d(\widetilde{B}_\tau, \widetilde{B}'_\tau)] \le 1 - \varepsilon. \tag{84}$$

Proof. Fix $x, x' \in B_R^{\mathsf{H}}$ and consider a reflection-type coupling of (76) on $[0, \tau]$: the two processes are driven by identical noise in the orthogonal complement of the instantaneous separation and reflected noise along the separation direction, so that once they meet they coalesce.

On B_R^{H} the drift map $x \mapsto -(x + \nabla_{\mathsf{H}} U(x))$ is globally Lipschitz with constant $1 + L_R$ by Lemma 7.9. Under reflection coupling, the separation norm $r_s = \|B_s - B'_s\|_{\mathsf{H}}$ dominates (up to a standard one-dimensional comparison argument) a one-dimensional diffusion with nondegenerate noise and bounded (on $[0, \tau]$) drift coefficient. Consequently, there exists $\varepsilon_0 = \varepsilon_0(R, \tau) \in (0, 1)$ such that the meeting probability satisfies

$$\mathbb{P}(B_s = B'_s \text{ for some } s \le \tau) \ge \varepsilon_0, \qquad \text{uniformly for } x, x' \in B_R^{\mathsf{H}}.$$

On the meeting event, $d(B_\tau, B'_\tau) = 0$; on its complement, $d(B_\tau, B'_\tau) \le 1$. Therefore

$$\mathbb{E}[d(B_\tau, B'_\tau)] \le (1 - \varepsilon_0) \cdot 1 + \varepsilon_0 \cdot 0 = 1 - \varepsilon_0.$$

This proves (84) with $\varepsilon = \varepsilon_0$ (and any choice of $\alpha > 0$). □

7.6 Quantitative Harris theorem and conclusion

We now state a convenient version of a weak Harris theorem in Wasserstein distance.

Theorem 7.11 (Weak Harris: drift + d-small set $\Rightarrow$ geometric ergodicity)**.** *Let P be a Markov kernel on a Polish space $\mathcal{X}$. Assume:*

(i) *(*Drift*) There exist $\lambda \in (0, 1)$ and $K < \infty$ and a measurable $V : \mathcal{X} \to [1, \infty)$ such that $PV \le \lambda V + K$.*

(ii) *(d-small set contraction) There exists a bounded measurable distance d on $\mathcal{X}$, a set $C = \{V \le R\}$ for some $R < \infty$, and $\varepsilon \in (0,1)$ such that for all $x, y \in C$,*

$$\mathcal{W}_d(P(x,\cdot), P(y,\cdot)) \le 1 - \varepsilon,$$

where $\mathcal{W}_d$ is the Wasserstein distance associated to d.

Then P admits a unique invariant probability measure π, and there exist constants $C < \infty$ and $\rho \in (0,1)$ such that for all probability measures μ with $\int V \, \mathrm{d}\mu < \infty$,

$$\mathcal{W}_d(\mu P^n, \pi) \le C \rho^n \int V \, \mathrm{d}\mu, \qquad n \in \mathbb{N}.$$

Corollary 7.12 (Geometric mixing for the boundary Langevin kernel)**.** *Fix $t > 0$ and $\tau > 0$ and let $P := P_\tau$ be the time-τ boundary Langevin kernel from Definition 7.3. Let V be (80) and d be (82). Then P admits the unique invariant measure $\pi_{t,\mathrm{Reg}}$ defined in (75), and there exist $C < \infty$ and $\rho \in (0,1)$ such that for all $x \in \mathsf{H}$,*

$$\mathcal{W}_d(P^n(x,\cdot), \pi_{t,\mathrm{Reg}}) \le C \rho^n V(x), \qquad n \in \mathbb{N}. \tag{85}$$

Proof. Apply Theorem 7.11 with $P = P_\tau$. The drift condition holds by Lemma 7.7. Choose R so that $C = \{V \le R\}$ contains B_R^{H} (up to the harmless additive constant in (80)). Then the d-small set contraction holds by Lemma 7.10. Uniqueness of the invariant measure follows, and by Proposition 7.5 the invariant measure is $\pi_{t,\mathrm{Reg}}$. ☐

8 Quantitative mixing consequences

This section converts the geometric ergodicity statement of Corollary 7.12 into concrete bounds on observables and correlations of the stationary boundary chain. These are the forms that will be used in later sections (in particular in the time-axis exponential clustering step).

Throughout, fix $t > 0$ and a time step $\tau > 0$. Let $P \equiv P_\tau$ denote the time-τ boundary Langevin kernel from Definition 7.3, let $\pi \equiv \pi_{t,\mathrm{Reg}}$ be its invariant law (75), let $V(x) = 1 + \|x\|_{\mathsf{H}}^2$ be the Lyapunov function (80), and let d be the bounded distance (82). Write P^n for the n-step kernel.

8.1 Wasserstein distance and Lipschitz test functions

Let $\mathcal{P}(\mathsf{H})$ denote the set of Borel probability measures on H. For a bounded measurable distance d on H, define the associated Wasserstein distance

$$\mathcal{W}_d(\mu,\nu) := \inf_{\Gamma \in \mathsf{Cpl}(\mu,\nu)} \int_{\mathsf{H}\times\mathsf{H}} d(x,y)\, \Gamma(\mathrm{d}x, \mathrm{d}y), \tag{86}$$

where $\mathsf{Cpl}(\mu,\nu)$ denotes the set of couplings of μ and ν.

For a function $f : \mathsf{H} \to \mathbb{R}$ define its d-Lipschitz seminorm by

$$\mathrm{Lip}_d(f) := \sup_{\substack{x,y\in\mathsf{H}\\ x\neq y}} \frac{|f(x) - f(y)|}{d(x,y)} \in [0,\infty]. \tag{87}$$

Lemma 8.1 (Dual bound for Lipschitz test functions)**.** *Let d be any bounded measurable distance on H and let $\mu, \nu \in \mathcal{P}(\mathsf{H})$. Then for every bounded measurable f with $\mathrm{Lip}_d(f) < \infty$,*

$$\left| \int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\nu \right| \le \mathrm{Lip}_d(f)\, \mathcal{W}_d(\mu,\nu). \tag{88}$$

Proof. Fix any coupling Γ of (μ, ν). Then, by definition of $\mathrm{Lip}_d(f)$,

$$\Big|\int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\nu\Big| = \Big|\int_{\mathsf{H}\times\mathsf{H}} (f(x)-f(y))\,\Gamma(\mathrm{d}x,\,\mathrm{d}y)\Big|$$
$$\le \int_{\mathsf{H}\times\mathsf{H}} |f(x)-f(y)|\,\Gamma(\mathrm{d}x,\,\mathrm{d}y)$$
$$\le \mathrm{Lip}_d(f)\int_{\mathsf{H}\times\mathsf{H}} d(x,y)\,\Gamma(\mathrm{d}x,\,\mathrm{d}y).$$

Taking the infimum over all couplings Γ yields (88). □

8.2 Convergence of Lipschitz observables from any initial condition

Corollary 7.12 gives geometric convergence in $\mathcal{W}_d$ from a point mass δ_x. We now convert it into an explicit bound on $P^n f(x) - \pi(f)$ for Lipschitz f.

Proposition 8.2 (Geometric convergence for d-Lipschitz observables)**.** *Assume (85) holds with constants $C < \infty$ and $\rho \in (0,1)$:*

$$\mathcal{W}_d(P^n(x,\cdot),\pi) \;\le\; C\,\rho^n\,V(x), \qquad n\in\mathbb{N}.$$

Then for every bounded measurable $f : \mathsf{H}\to\mathbb{R}$ with $\mathrm{Lip}_d(f) < \infty$ and every $x \in \mathsf{H}$,

$$\big|P^n f(x) - \pi(f)\big| \;\le\; \mathrm{Lip}_d(f)\,C\,\rho^n\,V(x), \qquad n\in\mathbb{N}. \tag{89}$$

More generally, for any initial distribution μ with $\int V\,\mathrm{d}\mu < \infty$,

$$\big|\mu P^n(f) - \pi(f)\big| \;\le\; \mathrm{Lip}_d(f)\,C\,\rho^n \int V\,\mathrm{d}\mu, \qquad n\in\mathbb{N}. \tag{90}$$

Proof. For (89), apply Lemma 8.1 with $\mu = P^n(x,\cdot)$ and $\nu = \pi$:

$$|P^n f(x) - \pi(f)| = \Big|\int f\,\mathrm{d}P^n(x,\cdot) - \int f\,\mathrm{d}\pi\Big| \le \mathrm{Lip}_d(f)\,\mathcal{W}_d(P^n(x,\cdot),\pi) \le \mathrm{Lip}_d(f)\,C\,\rho^n\,V(x).$$

For (90), integrate (88) against μ and use linearity:

$$|\mu P^n(f) - \pi(f)| = \Big|\int f\,\mathrm{d}(\mu P^n) - \int f\,\mathrm{d}\pi\Big| \le \mathrm{Lip}_d(f)\,\mathcal{W}_d(\mu P^n,\pi).$$

By convexity of $\mathcal{W}_d$ in its first argument and (85),

$$\mathcal{W}_d(\mu P^n,\pi) = \mathcal{W}_d\Big(\int P^n(x,\cdot)\,\mu(\mathrm{d}x),\pi\Big) \le \int \mathcal{W}_d(P^n(x,\cdot),\pi)\,\mu(\mathrm{d}x) \le C\,\rho^n\int V\,\mathrm{d}\mu,$$

which yields (90). □

8.3 Moment control under the invariant measure

We will need at least $\pi(V) < \infty$ to control correlations for bounded observables. This follows directly from the drift inequality.

Lemma 8.3 (Finite first moment under π)**.** *Assume the drift inequality (81) holds for P and V:*

$$PV \le \lambda V + K, \qquad \lambda\in(0,1),\;\; K<\infty.$$

Then $\pi(V) < \infty$, and in fact

$$\pi(V) \;\le\; \frac{K}{1-\lambda}. \tag{91}$$

Proof. Since π is invariant, $\pi(PV) = \pi(V)$. Integrate the drift inequality against π:

$$\pi(V) = \pi(PV) \le \lambda\pi(V) + K.$$

Rearrange to obtain $(1-\lambda)\pi(V) \le K$, which is (91). $\square$

We also record the corresponding bound for the chain started from any initial condition.

Lemma 8.4 (Uniform bound on $\mathbb{E}(V(B_n))$)**.** *Assume the drift inequality $PV \le \lambda V + K$. Let $(B_n)_{n\ge 0}$ be the discrete-time chain with kernel P. Then for every initial state $x \in \mathsf{H}$,*

$$\mathbb{E}_x[V(B_n)] \;\le\; \lambda^n V(x) + \frac{K}{1-\lambda}, \qquad n \in \mathbb{N}. \tag{92}$$

More generally, for an initial distribution μ with $\int V \,\mathrm{d}\mu < \infty$,

$$\mathbb{E}_\mu[V(B_n)] \;\le\; \lambda^n \int V \,\mathrm{d}\mu + \frac{K}{1-\lambda}.$$

Proof. Iterate the drift inequality:

$$P^n V \le \lambda^n V + K\sum_{j=0}^{n-1} \lambda^j = \lambda^n V + K\frac{1-\lambda^n}{1-\lambda}.$$

Evaluating at x gives (92) since $\mathbb{E}_x[V(B_n)] = P^n V(x)$. The general initial distribution statement follows by integrating against μ. $\square$

8.4 Exponential decorrelation in stationarity

We now convert (89) into a correlation bound for the stationary chain. Let $(B_n)_{n\ge 0}$ be the Markov chain with kernel P started from π (stationary). For measurable observables $F, G : \mathsf{H} \to \mathbb{R}$, define the stationary covariance

$$\mathrm{Cov}_\pi(F(B_0), G(B_n)) \;:=\; \mathbb{E}_\pi[F(B_0)G(B_n)] - \mathbb{E}_\pi[F(B_0)]\,\mathbb{E}_\pi[G(B_0)].$$

Proposition 8.5 (Covariance decay for bounded F and Lipschitz G)**.** *Assume (85) holds with constants (C, ρ) and assume the drift inequality holds with constants (λ, K). Let $G : \mathsf{H} \to \mathbb{R}$ satisfy $\mathrm{Lip}_d(G) < \infty$ and let $F : \mathsf{H} \to \mathbb{R}$ be bounded. Then for the stationary chain,*

$$\left|\mathrm{Cov}_\pi(F(B_0), G(B_n))\right| \;\le\; \|F\|_{L^\infty(\mathsf{H})}\,\mathrm{Lip}_d(G)\,C\,\rho^n\,\pi(V), \qquad n \in \mathbb{N}. \tag{93}$$

In particular, using Lemma 8.3, one may replace $\pi(V)$ by $K/(1-\lambda)$.

Proof. By the Markov property and stationarity,

$$\mathbb{E}_\pi\big[F(B_0)G(B_n)\big] = \mathbb{E}_\pi\big[F(B_0)\,(P^n G)(B_0)\big], \qquad \mathbb{E}_\pi[G(B_0)] = \pi(G).$$

Therefore

$$\mathrm{Cov}_\pi(F(B_0), G(B_n)) = \mathbb{E}_\pi\Big[F(B_0)\,\big((P^n G)(B_0) - \pi(G)\big)\Big]. \tag{94}$$

Taking absolute values and using $|F| \le \|F\|_\infty$,

$$\left|\mathrm{Cov}_\pi(F(B_0), G(B_n))\right| \le \|F\|_\infty\,\mathbb{E}_\pi\big[\,\left|(P^n G)(B_0) - \pi(G)\right|\,\big].$$

Apply Proposition 8.2 pointwise with $x = B_0$:

$$\left|(P^n G)(B_0) - \pi(G)\right| \le \mathrm{Lip}_d(G)\,C\,\rho^n\,V(B_0).$$

Take $\mathbb{E}_\pi[\cdot]$ and use $\mathbb{E}_\pi[V(B_0)] = \pi(V)$ to obtain (93). $\square$

Proposition 8.6 (Covariance decay with V-weighted observables)**.** *Assume* (85) *holds with constants* (C, ρ)*. Let* G *satisfy* $\mathrm{Lip}_d(G) < \infty$ *and let* F *be measurable such that* $\mathbb{E}_\pi[|F|\, V] < \infty$*. Then*

$$\left|\mathrm{Cov}_\pi(F(B_0), G(B_n))\right| \;\le\; \mathrm{Lip}_d(G)\, C\, \rho^n\, \mathbb{E}_\pi[|F|\, V], \qquad n \in \mathbb{N}. \tag{95}$$

Proof. Starting from (94) and using $|ab| \le |a||x|$,

$$|\mathrm{Cov}_\pi(F(B_0), G(B_n))| \le \mathbb{E}_\pi\Big[|F(B_0)|\, \big|(P^n G)(B_0) - \pi(G)\big|\Big].$$

Apply Proposition 8.2 pointwise with $x = B_0$ and then take expectation:

$$\big|(P^n G)(B_0) - \pi(G)\big| \le \mathrm{Lip}_d(G)\, C\, \rho^n\, V(B_0).$$

Thus

$$|\mathrm{Cov}_\pi(F(B_0), G(B_n))| \le \mathrm{Lip}_d(G)\, C\, \rho^n\, \mathbb{E}_\pi[|F|\, V],$$

which is (95). ☐

8.5 Mixing bounds for multi-time observables

For later applications it is useful to handle observables depending on several consecutive steps. We formulate this in a way that is compatible with the constants ledger.

Let $m \ge 0$ and let $\mathbf{F} : \mathsf{H}^{m+1} \to \mathbb{R}$ and $\mathbf{G} : \mathsf{H}^{m+1} \to \mathbb{R}$ be bounded measurable. Define the windowed observables

$$F_m \coloneqq \mathbf{F}(B_0, B_1, \ldots, B_m), \qquad G_{m,n} \coloneqq \mathbf{G}(B_n, B_{n+1}, \ldots, B_{n+m}), \qquad n \ge m+1.$$

The correct Lipschitz quantity for $\mathbf{G}$ in this context is the Lipschitz seminorm of the *reduced* single-site observable obtained by integrating out the window dynamics.

Definition 8.7 (Reduced Lipschitz window seminorm)**.** Fix $m \ge 0$. For bounded measurable $\mathbf{G} : \mathsf{H}^{m+1} \to \mathbb{R}$, define the reduced function $g_{\mathbf{G}} : \mathsf{H} \to \mathbb{R}$ by

$$g_{\mathbf{G}}(x) \;\coloneqq\; \mathbb{E}_x\big[\mathbf{G}(B_0, B_1, \ldots, B_m)\big], \tag{96}$$

where $(B_k)_{k\ge 0}$ is the Markov chain with kernel P started from $B_0 = x$. Define

$$\mathrm{Lip}_d^{(0)}(\mathbf{G}) \;\coloneqq\; \mathrm{Lip}_d(g_{\mathbf{G}}) \in [0, \infty]. \tag{97}$$

Proposition 8.8 (Decay of correlations for separated windows)**.** *Assume* (85) *holds with constants* (C, ρ) *and assume* $\pi(V) < \infty$*. Let* $\mathbf{F}, \mathbf{G}$ *be bounded and suppose* $\mathrm{Lip}_d^{(0)}(\mathbf{G}) < \infty$ *in the sense of Definition 8.7. Then for the stationary chain and for all* $n \ge m+1$*,*

$$\left|\mathrm{Cov}_\pi(F_m, G_{m,n})\right| \;\le\; \|\mathbf{F}\|_\infty\, \mathrm{Lip}_d^{(0)}(\mathbf{G})\, C\, \rho^{n-m}\, \pi(V). \tag{98}$$

Proof. By the Markov property, conditional on B_m the future chain $(B_{m+\ell})_{\ell\ge 0}$ is a Markov chain started from B_m. In particular, for $n \ge m+1$,

$$\mathbb{E}_\pi\big[G_{m,n} \,\big|\, B_m\big] = (P^{n-m} g_{\mathbf{G}})(B_m),$$

where $g_{\mathbf{G}}$ is (96). Since π is invariant, $\mathbb{E}_\pi[G_{m,n}] = \pi(g_{\mathbf{G}})$. Therefore

$$\mathrm{Cov}_\pi(F_m, G_{m,n}) = \mathbb{E}_\pi\Big[F_m\Big((P^{n-m} g_{\mathbf{G}})(B_m) - \pi(g_{\mathbf{G}})\Big)\Big].$$

Taking absolute values and using $|F_m| \le \|\mathbf{F}\|_\infty$,

$$\big|\mathrm{Cov}_\pi(F_m, G_{m,n})\big| \le \|\mathbf{F}\|_\infty \, \mathbb{E}_\pi\Big[\Big|(P^{n-m} g_{\mathbf{G}})(B_m) - \pi(g_{\mathbf{G}})\Big|\Big].$$

Apply Proposition 8.2 pointwise with $f = g_{\mathbf{G}}$ and $x = B_m$:

$$\Big|(P^{n-m} g_{\mathbf{G}})(B_m) - \pi(g_{\mathbf{G}})\Big| \le \mathrm{Lip}_d(g_{\mathbf{G}})\, C\, \rho^{n-m}\, V(B_m).$$

Take $\mathbb{E}_\pi[\cdot]$ and use stationarity $\mathbb{E}_\pi[V(B_m)] = \pi(V)$ to obtain (98). $\square$

8.6 Regulator-uniformity bookkeeping

The constants (C, ρ) in (85) and the drift constants (λ, K) enter the correlation bounds (93), (95), and (98) only through the combination $C\rho^n$ and the moment $\pi(V) \le K/(1-\lambda)$. Therefore, any regulator-uniform control on (C, ρ, λ, K) immediately yields regulator-uniform mixing and decorrelation bounds for the class of bounded d-Lipschitz observables, and for window observables $\mathbf{G}$ with controlled reduced seminorm $\mathrm{Lip}_d^{(0)}(\mathbf{G})$.

In later sections we will apply these bounds to bounded Lipschitz cylinder functionals of the boundary trace, constructed from quasi-local response kernels (Section 4) and their moment bounds (Section 5).

9 Time-axis exponential clustering and reflection positivity for the boundary process

This section packages the output of Sections 7–8 into an Osterwalder–Schrader (OS) style statement for the *stationary boundary process* generated by the Markov kernel $P = P_\tau$. Concretely, we construct the bi-infinite path measure, prove a reflection-positivity identity from reversibility, and then prove an time-axis exponential clustering *exponential clustering* estimate for a large, explicitly controlled class of cylinder observables.

9.1 The stationary bi-infinite boundary process

Fix $t > 0$ and $\tau > 0$ and recall:

- $\mathsf{H} = \mathcal{B}_{L,\Lambda}$ is the finite-dimensional boundary state space at regulator level Reg, equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathsf{H}}$ (74);
- $P = P_\tau$ is the Markov kernel on H from Definition 7.3;
- $\pi = \pi_{t,\mathrm{Reg}}$ is the unique invariant law (75), and P is reversible w.r.t. π (Proposition 7.5);
- the mixing estimates of Section 8 hold in the bounded distance d (82).

Let $\Omega := \mathsf{H}^{\mathbb{Z}}$ be the path space and let $\mathcal{F}$ be the product Borel σ-algebra. For $n \in \mathbb{Z}$ let $\omega \mapsto \omega_n$ be the coordinate maps.

Definition 9.1 (Stationary path measure). Let $\mathbb{P}_\pi$ denote the law on $(\Omega, \mathcal{F})$ of the stationary Markov chain $(B_n)_{n\in\mathbb{Z}}$ with transition kernel P and one-time marginal π.

Lemma 9.2 (Existence and uniqueness of $\mathbb{P}_\pi$)**.** *There exists a unique probability measure $\mathbb{P}_\pi$ on Ω such that: for every $m \le n$ and every bounded measurable $\Phi : \mathsf{H}^{n-m+1} \to \mathbb{R}$,*

$$\mathbb{E}_{\mathbb{P}_\pi}[\Phi(B_m, \dots, B_n)] = \int_{\mathsf{H}} \pi(\mathrm{d}x_m) \int_{\mathsf{H}} P(x_m, \mathrm{d}x_{m+1}) \cdots \int_{\mathsf{H}} P(x_{n-1}, \mathrm{d}x_n)\, \Phi(x_m, \dots, x_n). \tag{99}$$

Moreover, $\mathbb{P}_\pi$ is invariant under the left shift $\mathsf{S} : \Omega \to \Omega$, $(\mathsf{S}\omega)_n = \omega_{n+1}$.

Proof. Define the finite-dimensional distributions by (99). Consistency in the Kolmogorov sense holds because integrating out an intermediate coordinate x_k simply composes kernels, and integrating out the endpoints uses the invariance $\pi P = \pi$. Since H is a Polish space, Kolmogorov's extension theorem yields existence of a measure $\mathbb{P}_\pi$ on Ω with these marginals, and uniqueness follows because cylinder sets generate $\mathcal{F}$.

Shift invariance follows by direct computation of cylinder expectations: for any cylinder observable depending on $(B_m, \dots, B_n)$, the joint distribution of $(B_{m+1}, \dots, B_{n+1})$ under (99) agrees with that of $(B_m, \dots, B_n)$ because π is invariant and the kernel is time-homogeneous. $\square$

9.2 Time reflection and reflection positivity

Define the time reflection map $\Theta : \Omega \to \Omega$ by

$$(\Theta\omega)_n \;\coloneqq\; \omega_{-n-1}, \qquad n \in \mathbb{Z}. \tag{100}$$

This choice reflects about the "midpoint" between times -1 and 0 (the standard discrete-time OS choice).

Let $\mathcal{F}_+$ be the σ-algebra generated by $\{\omega_n : \; n \ge 0\}$, and let $\mathcal{F}_-$ be generated by $\{\omega_n : \; n \le -1\}$. For any bounded measurable $F : \Omega \to \mathbb{R}$, define $(\Theta F)(\omega) \coloneqq F(\Theta\omega)$.

Definition 9.3 (Positive-time cylinder algebra)**.** Let $\mathcal{A}_+$ be the set of bounded cylinder functions F that are $\mathcal{F}_+$-measurable, i.e. depend only on finitely many coordinates $(\omega_0, \dots, \omega_m)$ for some $m \ge 0$.

Introduce the Markov operator T on bounded measurable $f : \mathsf{H} \to \mathbb{R}$ by

$$(Tf)(x) \;\coloneqq\; \int_{\mathsf{H}} f(x')\, P(x, \mathrm{d}x'). \tag{101}$$

By Proposition 7.5, T extends to a self-adjoint contraction on $L^2(\pi)$:

$$\langle f, Tg \rangle_{L^2(\pi)} = \langle Tf, g \rangle_{L^2(\pi)}, \qquad \|Tf\|_{L^2(\pi)} \le \|f\|_{L^2(\pi)}. \tag{102}$$

For $m \ge 0$ and bounded measurable $f_0, \dots, f_m : \mathsf{H} \to \mathbb{R}$ define the positive-time cylinder

$$F(\omega) \;\coloneqq\; \prod_{j=0}^{m} f_j(\omega_j). \tag{103}$$

Define the corresponding "transfer" map $\Phi : \mathcal{A}_+ \to L^2(\pi)$ on such monomials by

$$\Phi(F) \;\coloneqq\; f_0\, T\big(f_1\, T(\cdots T(f_m) \cdots)\big), \tag{104}$$

where products are pointwise and T acts as (101). Extend Φ to finite linear combinations by linearity.

Lemma 9.4 (Reflection positivity identity)**.** *For all $F, G \in \mathcal{A}_+$,*

$$\mathbb{E}_{\mathbb{P}_\pi}\big[F\,(\Theta G)\big] \;=\; \langle \Phi(F), \Phi(G)\rangle_{L^2(\pi)}. \tag{105}$$

In particular,

$$\mathbb{E}_{\mathbb{P}_\pi}\big[F\,(\Theta F)\big] \;\geq\; 0, \qquad F \in \mathcal{A}_+. \tag{106}$$

Proof. It suffices to verify (105) for monomials of the form (103). Let

$$F(\omega) = \prod_{i=0}^{m} f_i(\omega_i), \qquad G(\omega) = \prod_{j=0}^{n} g_j(\omega_j),$$

with bounded measurable f_i, g_j. Then $(\Theta G)(\omega) = \prod_{j=0}^{n} g_j(\omega_{-j-1})$.

By stationarity, the random variable B_{-1} has law π. Conditional on $B_{-1} = x$, the future block $(B_0, \ldots, B_m)$ is a Markov chain started at x and evolved forward with kernel P, so iterated conditioning yields

$$\mathbb{E}_{\mathbb{P}_\pi}\big[F \,\big|\, B_{-1} = x\big] = \Phi(F)(x).$$

Similarly, by reversibility (detailed balance), the time-reversed chain has the same transition kernel P, so conditional on $B_{-1} = x$ the past block $(B_{-n-1}, \ldots, B_{-1})$ evolved backward is also governed by P. Thus

$$\mathbb{E}_{\mathbb{P}_\pi}\big[\Theta G \,\big|\, B_{-1} = x\big] = \Phi(G)(x).$$

Using conditional independence of past and future given B_{-1} for a Markov chain, we obtain

$$\mathbb{E}_{\mathbb{P}_\pi}\big[F\,(\Theta G)\big] = \int_{\mathsf{H}} \pi(\,\mathrm{d}x)\,\Phi(F)(x)\,\Phi(G)(x) = \langle \Phi(F), \Phi(G)\rangle_{L^2(\pi)},$$

which is (105). Taking $G = F$ yields (106). $\square$

9.3 Boundary OS pre-Hilbert space and transfer operator

Reflection positivity (106) defines a positive semidefinite bilinear form on $\mathcal{A}_+$.

Definition 9.5 (OS inner product on $\mathcal{A}_+$)**.** For $F, G \in \mathcal{A}_+$ define

$$(F, G)_{\mathrm{OS}} \;:=\; \mathbb{E}_{\mathbb{P}_\pi}\big[F\,(\Theta G)\big]. \tag{107}$$

Let $\mathcal{N} := \{F \in \mathcal{A}_+ :\; (F, F)_{\mathrm{OS}} = 0\}$ be the null space. Define the pre-Hilbert space $\mathcal{D} := \mathcal{A}_+/\mathcal{N}$ with inner product induced by (107), and let $\mathcal{H}_{\mathrm{OS}}$ be its Hilbert space completion.

Remark 9.6. Lemma 9.4 shows that the map $\Phi : \mathcal{A}_+ \to L^2(\pi)$ is an isometry modulo the null space: $(F, G)_{\mathrm{OS}} = \langle \Phi(F), \Phi(G)\rangle_{L^2(\pi)}$. In particular, $\mathcal{H}_{\mathrm{OS}}$ can be identified with the closure of $\Phi(\mathcal{A}_+)$ in $L^2(\pi)$.

Time translations on the path space induce contractions on $\mathcal{H}_{\mathrm{OS}}$. Let S be the left shift on Ω and note S maps $\mathcal{F}_+$ to itself.

Definition 9.7 (Time translation operator)**.** Define $\mathcal{T} : \mathcal{A}_+ \to \mathcal{A}_+$ by $(\mathcal{T}F)(\omega) := F(\mathsf{S}\omega) = F(\omega_1, \omega_2, \ldots)$. Then $\mathcal{T}$ descends to a contraction on $\mathcal{H}_{\mathrm{OS}}$, still denoted $\mathcal{T}$.

Lemma 9.8 (Contraction and identification with T)**.** *Under the identification of Remark 9.6, the operator $\mathcal{T}$ corresponds to the Markov operator T on $L^2(\pi)$: for all $F \in \mathcal{A}_+$,*

$$\Phi(\mathcal{T}F) \;=\; T\Phi(F). \tag{108}$$

In particular, $\|\mathcal{T}\|_{\mathcal{H}_{\mathrm{OS}} \to \mathcal{H}_{\mathrm{OS}}} \leq 1$.

Proof. It suffices to check (108) on monomials $F(\omega) = \prod_{j=0}^{m} f_j(\omega_j)$. Then $(\mathcal{T}F)(\omega) = \prod_{j=0}^{m} f_j(\omega_{j+1})$. By definition (104),

$$\Phi(F) = f_0\, T\big(f_1\, T(\cdots T(f_m)\cdots)\big), \qquad \Phi(\mathcal{T}F) = T\big(f_0\, T(f_1\, T(\cdots T(f_m)\cdots))\big) = T\Phi(F).$$

This proves (108). Since T is an $L^2(\pi)$ contraction, the corresponding OS operator is a contraction as well. ☐

9.4 Time-axis exponential clustering from Harris mixing

We now state and prove the time-axis exponential clustering property for the boundary process, in a form adapted to the quantitative bounds of Section 8.

Definition 9.9 (Admissible boundary observables)**.** Let $\mathcal{L}_d$ denote the set of measurable functions $f : \mathsf{H} \to \mathbb{R}$ such that:

(i) f is bounded, i.e. $\|f\|_{L^\infty(\mathsf{H})} < \infty$;

(ii) f is d-Lipschitz, i.e. $\mathrm{Lip}_d(f) < \infty$.

Theorem 9.10 (Boundary time-axis exponential clustering: exponential clustering for separated time blocks)**.** *Assume the geometric mixing estimate (85) holds: there exist constants $C < \infty$ and $\rho \in (0,1)$ such that $\mathcal{W}_d(P^n(x,\cdot),\pi) \le C\rho^n V(x)$ for all $x \in \mathsf{H}$ and $n \in \mathbb{N}$. Assume also $\pi(V) < \infty$ (which follows from the drift inequality by Lemma 8.3).*

Fix $m \ge 0$. Let $\mathbf{F} : \mathsf{H}^{m+1} \to \mathbb{R}$ be bounded measurable and let $\mathbf{G} : \mathsf{H}^{m+1} \to \mathbb{R}$ be bounded measurable with $\mathrm{Lip}_d^{(0)}(\mathbf{G}) < \infty$ in the sense of Definition 8.7. Define the stationary window observables

$$F_m \coloneqq \mathbf{F}(B_0, \ldots, B_m), \qquad G_{m,n} \coloneqq \mathbf{G}(B_n, \ldots, B_{n+m}), \qquad n \ge m+1.$$

Then the stationary covariance satisfies the exponential clustering bound

$$\big|\mathrm{Cov}_\pi(F_m, G_{m,n})\big| \;\le\; \|\mathbf{F}\|_\infty\, \mathrm{Lip}_d^{(0)}(\mathbf{G})\, C\, \rho^{\,n-m}\, \pi(V), \qquad n \ge m+1. \tag{109}$$

In particular, for $f \in L^\infty(\mathsf{H})$ and $g \in \mathcal{L}_d$,

$$\big|\mathrm{Cov}_\pi(f(B_0), g(B_n))\big| \;\le\; \|f\|_\infty\, \mathrm{Lip}_d(g)\, C\, \rho^{\,n}\, \pi(V), \qquad n \in \mathbb{N}. \tag{110}$$

Proof. Bound (109) is exactly Proposition 8.8. Specialising (109) to $m = 0$ yields (110). ☐

Remark 9.11 (Correlation length and "mass parameter")*.* The decay rate $\rho \in (0,1)$ defines a canonical inverse correlation length in the discrete time variable n. If one declares the physical Euclidean time increment to be τ per step (as in our kernel P_τ), then the exponential clustering rate (110) corresponds to

$$\rho^n = e^{-n|\log\rho|} = e^{-(n\tau)\,m_*}, \qquad m_* \coloneqq \frac{|\log\rho|}{\tau}.$$

In this paper we interpret m_* as the *boundary time-clustering rate* produced by Harris mixing.

9.5 OS axioms for the boundary process: summary

We summarise the OS properties obtained for the boundary process $\mathbb{P}_\pi$ on $\Omega = \mathsf{H}^{\mathbb{Z}}$. Let $\mathcal{A}_+$ be the positive-time cylinder algebra (Definition 9.3) and Θ the time reflection (100).

Proposition 9.12 (Boundary OS package)**.** *At each fixed regulator level* Reg*, the stationary boundary process* $(B_n)_{n\in\mathbb{Z}}$ *under* $\mathbb{P}_\pi$ *satisfies:*

(i) *(*OS0: regularity on cylinder algebra*) Every* $F \in \mathcal{A}_+$ *is bounded and measurable, hence* $F \in L^p(\mathbb{P}_\pi)$ *for all* $p \in [1, \infty)$*.*

(ii) *(*OS1: time-translation invariance*)* $\mathbb{P}_\pi$ *is invariant under the shift* S *(Lemma 9.2).*

(iii) *(*OS2: reflection positivity*) For all* $F \in \mathcal{A}_+$*,* $\mathbb{E}_{\mathbb{P}_\pi}[F(\Theta F)] \geq 0$ *(Lemma 9.4).*

(iv) *(*OS4: exponential clustering in the time direction*) For admissible cylinder observables as in Theorem 9.10, time-separated covariances decay exponentially with rate* ρ *uniformly as in* (109)*–*(110)*.*

Proof. Items (i)–(iii) are immediate from the constructions and Lemma 9.4. Item (iv) is Theorem 9.10. □

9.6 Regulator-uniformity statement

Finally, we spell out the uniformity implication that will be used downstream.

Corollary 9.13 (Regulator-uniform time-axis exponential clustering on admissible observables)**.** *Suppose there exist constants* $C < \infty$*,* $\rho \in (0, 1)$*, and* $M < \infty$ *such that for all regulator levels* Reg*:*

(i) $\mathcal{W}_d(P^n(x, \cdot), \pi) \leq C\rho^n V(x)$ *for all* x *and* n*;*

(ii) $\pi(V) \leq M$*.*

Then the exponential clustering bounds (109)*–*(110) *hold with the* same *constants* C, ρ, M *for all* Reg*. Equivalently, all admissible stationary covariances decay at least as fast as* $e^{-m_*(n\tau)}$ *with* $m_* = |\log \rho|/\tau$ *uniformly in* Reg*.*

Proof. Immediate from Theorem 9.10 by inserting the uniform bounds. □

Remark 9.14 (Scope of Section 9). Section 9 establishes OS-style reflection positivity and time-axis exponential clustering for the *boundary process* induced by the regulator-level kernel P_τ and its invariant measure π. In subsequent sections, these bounds are transported to gauge-invariant slab observables via the quasi-local response representation and uniform moment controls developed earlier.

10 Transport from boundary mixing to slab observables

Sections 7–9 establish a reversible stationary boundary process $(B_n)_{n\in\mathbb{Z}}$ on H with reflection positivity and time-axis exponential clustering in the discrete time index n. This section shows how to *lift* those bounds to observables of the regulated Yang–Mills field on slabs of thickness t.

The mechanism is:

(i) define a "stacked" field on $\mathbb{R} \times \mathbb{T}^3_L$ by gluing independent slab interiors conditional on consecutive boundary states;

(ii) show that slab-local observables reduce to bounded functions of finitely many boundary variables;

(iii) control the relevant reduced Lipschitz seminorms via response bounds, and apply the mixing bounds of Section 8.

All statements are at fixed regulator level Reg; the uniformity discussion at the end records precisely what must be uniform in Reg to obtain regulator-uniform clustering.

10.1 One-slab configuration space and conditional interior laws

Fix $t > 0$ and $L > 0$, and write $S_{t,L} = [0, t] \times \mathbb{T}^3_L$. At regulator level Reg (e.g. Fourier cutoff Λ plus any gauge-fixing regularisation), let $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ denote the finite-dimensional configuration space of regulated gauge fields on $S_{t,L}$ (as defined in Section 4). Let

$$\mathrm{Tr}_- : \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} \to \mathsf{H}, \qquad \mathrm{Tr}_+ : \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} \to \mathsf{H}$$

be the boundary trace maps at times 0 and t respectively (also fixed in Section 4).

For $(x^-, x^+) \in \mathsf{H} \times \mathsf{H}$, define the (nonempty) affine subspace of fields with prescribed endpoints:

$$\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}(x^-, x^+) \;:=\; \{A \in \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} :\; \mathrm{Tr}_-(A) = x^-,\; \mathrm{Tr}_+(A) = x^+\}.$$

Let $\mathbf{m}_{t,\mathrm{Reg}}(\,\mathrm{d}A)$ be the reference Gaussian measure on $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ used in Section 4 to define the interacting slab law, and let $S_{t,\mathrm{Reg}}(A)$ be the regulated (gauge-fixed) Euclidean action functional. Define the unnormalised density

$$\mathbf{w}_{t,\mathrm{Reg}}(\,\mathrm{d}A) \;:=\; e^{-S_{t,\mathrm{Reg}}(A)}\, \mathbf{m}_{t,\mathrm{Reg}}(\,\mathrm{d}A).$$

Lemma 10.1 (Disintegration into endpoint kernel and interior conditional laws)**.** *There exist:*

(i) *a probability kernel* $\mathbf{K}_{t,\mathrm{Reg}}$ *from* $\mathsf{H} \times \mathsf{H}$ *to* $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$*, written* $\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}(\,\mathrm{d}A)$ *and supported on* $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}(x^-, x^+)$*;*

(ii) *a finite measure* $\boldsymbol{\kappa}_{t,\mathrm{Reg}}(\,\mathrm{d}x^-,\,\mathrm{d}x^+)$ *on* $\mathsf{H} \times \mathsf{H}$*;*

such that for every bounded measurable $\Psi : \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} \to \mathbb{R}$*,*

$$\int_{\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}} \Psi(A)\, \mathbf{w}_{t,\mathrm{Reg}}(\,\mathrm{d}A) = \int_{\mathsf{H}\times\mathsf{H}} \Big(\int_{\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}} \Psi(A)\, \mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}(\,\mathrm{d}A) \Big)\, \boldsymbol{\kappa}_{t,\mathrm{Reg}}(\,\mathrm{d}x^-,\,\mathrm{d}x^+). \tag{111}$$

Moreover, $\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}$ *is uniquely defined for* $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$*-a.e.* (x^-, x^+)*.*

Proof. Because $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ and H are finite-dimensional (hence Polish), the map $A \mapsto (\mathrm{Tr}_-(A), \mathrm{Tr}_+(A))$ is Borel. Define $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$ as the pushforward of $\mathbf{w}_{t,\mathrm{Reg}}$ under this endpoint map:

$$\boldsymbol{\kappa}_{t,\mathrm{Reg}}(E) \;:=\; \mathbf{w}_{t,\mathrm{Reg}}\Big(\{A :\; (\mathrm{Tr}_-(A), \mathrm{Tr}_+(A)) \in E\}\Big), \qquad E \subset \mathsf{H} \times \mathsf{H} \text{ Borel}.$$

Since $\mathbf{w}_{t,\mathrm{Reg}}$ is finite (at fixed regulator level), $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$ is finite.

By the disintegration theorem for finite Borel measures on Polish spaces, there exists a measurable family of conditional probability measures $\{\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}\}$ such that (111) holds for all bounded measurable Ψ, and $\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}$ is supported on the fibre $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}(x^-, x^+)$ for $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$-a.e. (x^-, x^+). Uniqueness holds up to $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$-null sets. ∎

10.2 Stacked slab field driven by the boundary chain

Let $(B_n)_{n\in\mathbb{Z}}$ be the stationary boundary chain under $\mathbb{P}_\pi$ from Definition 9.1, with $\pi = \pi_{t,\mathrm{Reg}}$ and kernel $P = P_\tau$. We now build a random field on the bi-infinite time axis by gluing independent slab interiors conditional on consecutive boundary states.

For each $n \in \mathbb{Z}$, let $S^{(n)}_{t,L} := [nt,(n+1)t] \times \mathbb{T}^3_L$ and let $\mathsf{X}^{(n)}_{t,\mathrm{Reg}}$ be a copy of $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ representing regulated fields on $S^{(n)}_{t,L}$ (via time translation). Let $\mathbf{Q}^{x^-,x^+,(n)}_{t,\mathrm{Reg}}$ denote the corresponding translated conditional law on $\mathsf{X}^{(n)}_{t,\mathrm{Reg}}$.

Definition 10.2 (Stacked field measure)**.** Define a probability measure $\mathbf{P}^{\mathrm{stack}}_{t,\mathrm{Reg}}$ on the product space

$$\Omega_{\mathrm{stack}} \;:=\; \mathsf{H}^{\mathbb{Z}} \times \prod_{n\in\mathbb{Z}} \mathsf{X}^{(n)}_{t,\mathrm{Reg}}$$

as follows:

(i) sample the boundary path $(B_n)_{n\in\mathbb{Z}}$ with law $\mathbb{P}_\pi$;

(ii) conditional on (B_n), sample independent slab interiors $A^{(n)} \in \mathsf{X}^{(n)}_{t,\mathrm{Reg}}$ with

$$A^{(n)} \;\sim\; \mathbf{Q}^{B_n,B_{n+1},(n)}_{t,\mathrm{Reg}}, \qquad \text{independently for different } n \in \mathbb{Z}. \tag{112}$$

Lemma 10.3 (Conditional independence and factorisation)**.** *Let $\mathcal{O}_n$ be bounded measurable functions on $\mathsf{X}^{(n)}_{t,\mathrm{Reg}}$ depending on only finitely many indices n. Then under $\mathbf{P}^{\mathrm{stack}}_{t,\mathrm{Reg}}$,*

$$\mathbb{E}\Big[\prod_{n\in\mathbb{Z}} \mathcal{O}_n(A^{(n)})\Big] = \mathbb{E}_{\mathbb{P}_\pi}\Big[\prod_{n\in\mathbb{Z}} \mathcal{R}\mathcal{O}_n(B_n,B_{n+1})\Big], \tag{113}$$

where the product is over the finitely many n for which $\mathcal{O}_n \not\equiv 1$, and where

$$(\mathcal{R}\mathcal{O})(x^-,x^+) \;:=\; \int_{\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}} \mathcal{O}(A)\, \mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}(\,\mathrm{d}A). \tag{114}$$

Proof. By construction, conditional on the boundary path (B_n), the random variables $A^{(n)}$ are independent with laws $\mathbf{Q}^{B_n,B_{n+1},(n)}_{t,\mathrm{Reg}}$. Hence, conditioning on (B_n) and using Fubini's theorem,

$$\mathbb{E}\Big[\prod_n \mathcal{O}_n(A^{(n)}) \;\Big|\; (B_k)_{k\in\mathbb{Z}}\Big] = \prod_n \int \mathcal{O}_n(A)\, \mathbf{Q}^{B_n,B_{n+1},(n)}_{t,\mathrm{Reg}}(\,\mathrm{d}A).$$

The time translation in (n) does not change the value of the integral, so the right-hand side equals $\prod_n (\mathcal{R}\mathcal{O}_n)(B_n,B_{n+1})$. Taking expectation over $\mathbb{P}_\pi$ gives (113). $\square$

10.3 Lipschitz control of reduced slab observables via response bounds

To apply the boundary mixing bounds (Section 8), we must control the reduced Lipschitz seminorms of functions obtained from one-slab observables by conditioning and reduction.

We begin with a deterministic lemma relating H-Lipschitz bounds to d-Lipschitz bounds.

Lemma 10.4 ($\|\cdot\|_{\mathsf{H}}$-Lipschitz implies d-Lipschitz)**.** *Let d be as in (82): $d(x,y) = 1 \wedge (\alpha\|x-y\|_{\mathsf{H}})$. Let $f : \mathsf{H} \to \mathbb{R}$ be bounded and globally Lipschitz w.r.t. $\|\cdot\|_{\mathsf{H}}$, i.e. $|f(x)-f(y)| \le L_{\mathsf{H}}\|x-y\|_{\mathsf{H}}$ for all x,y. Then f is d-Lipschitz with*

$$\mathrm{Lip}_d(f) \;\le\; \max\Big\{\frac{L_{\mathsf{H}}}{\alpha},\, 2\|f\|_{L^\infty(\mathsf{H})}\Big\}. \tag{115}$$

Proof. If $d(x,y)=\alpha\|x-y\|_{\mathsf{H}}$, then $|f(x)-f(y)|/d(x,y)\le L_{\mathsf{H}}/\alpha$. If $d(x,y)=1$, then $|f(x)-f(y)|/d(x,y)\le 2\|f\|_\infty$. Taking the supremum over $x\neq y$ gives (115). $\square$

Since the slab measure depends on endpoints (x^-,x^+), the reduced observable is a function on $\mathsf{H}\times\mathsf{H}$.

Definition 10.5 (Admissible one-slab observables)**.** A bounded measurable function $\mathcal{O}:\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}\to\mathbb{R}$ is called *endpoint-admissible* if the map

$$F_{\mathcal{O}}(x^-,x^+)\ :=\ (\mathcal{R}\mathcal{O})(x^-,x^+)=\int \mathcal{O}(A)\,\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}(\,\mathrm{d}A)$$

is C^1 in the endpoint variables (at least on balls), with derivatives given by the response identities of Section 4.

Lemma 10.6 (Endpoint response identity and gradient bound)**.** *Let $\mathcal{O}$ be endpoint-admissible in the sense of Definition 10.5. Assume that for each (x^-,x^+) there exist H-valued random variables $\mathscr{J}^-_{t,\mathrm{Reg}}(x^-,x^+;A)$ and $\mathscr{J}^+_{t,\mathrm{Reg}}(x^-,x^+;A)$ on $(\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}},\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}})$ such that:*

(i) *for every $h\in\mathsf{H}$,*

$$D_{x^+}F_{\mathcal{O}}(x^-,x^+)[h]=\mathrm{Cov}_{\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}}\Big(\mathcal{O}(A),\,\langle\mathscr{J}^+_{t,\mathrm{Reg}}(x^-,x^+;A),h\rangle_{\mathsf{H}}\Big),\tag{116}$$

and likewise

$$D_{x^-}F_{\mathcal{O}}(x^-,x^+)[h]=\mathrm{Cov}_{\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}}\Big(\mathcal{O}(A),\,\langle\mathscr{J}^-_{t,\mathrm{Reg}}(x^-,x^+;A),h\rangle_{\mathsf{H}}\Big);\tag{117}$$

(ii) *there exists a constant $M_{\mathscr{J}}(t,R)<\infty$ such that for all $\|x^\pm\|_{\mathsf{H}}\le R$,*

$$\mathbb{E}_{\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}}\big[\|\mathscr{J}^\pm_{t,\mathrm{Reg}}(x^-,x^+;A)\|_{\mathsf{H}}\big]\ \le\ M_{\mathscr{J}}(t,R),\tag{118}$$

(with the same bound for $\mathscr{J}^-$ and $\mathscr{J}^+$).

Then for all $\|x^\pm\|_{\mathsf{H}}\le R$,

$$\big\|\nabla_{x^\pm}F_{\mathcal{O}}(x^-,x^+)\big\|_{\mathsf{H}}\ \le\ 2\,\|\mathcal{O}\|_{L^\infty(\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}})}\,M_{\mathscr{J}}(t,R).\tag{119}$$

Proof. Fix $\|x^\pm\|_{\mathsf{H}}\le R$ and $h\in\mathsf{H}$. We treat x^+; the x^- case is identical using (117). By (116) and the definition of covariance,

$$D_{x^+}F_{\mathcal{O}}(x^-,x^+)[h]=\mathbb{E}[\mathcal{O}\,\langle\mathscr{J}^+,h\rangle_{\mathsf{H}}]-\mathbb{E}[\mathcal{O}]\,\mathbb{E}[\langle\mathscr{J}^+,h\rangle_{\mathsf{H}}],$$

where expectations are under $\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}$. Hence

$$\begin{aligned}\big|D_{x^+}F_{\mathcal{O}}(x^-,x^+)[h]\big|&\le\mathbb{E}\big[|\mathcal{O}|\,|\langle\mathscr{J}^+,h\rangle_{\mathsf{H}}|\big]+\mathbb{E}[|\mathcal{O}|]\,\mathbb{E}\big[|\langle\mathscr{J}^+,h\rangle_{\mathsf{H}}|\big]\\&\le 2\|\mathcal{O}\|_\infty\,\mathbb{E}\big[|\langle\mathscr{J}^+,h\rangle_{\mathsf{H}}|\big]\\&\le 2\|\mathcal{O}\|_\infty\,\|h\|_{\mathsf{H}}\,\mathbb{E}\big[\|\mathscr{J}^+\|_{\mathsf{H}}\big].\end{aligned}$$

Using (118) gives

$$\big|D_{x^+}F_{\mathcal{O}}(x^-,x^+)[h]\big|\le 2\|\mathcal{O}\|_\infty\,M_{\mathscr{J}}(t,R)\,\|h\|_{\mathsf{H}}.$$

By Riesz representation on H, this implies (119). $\square$

Proposition 10.7 (Local d-Lipschitz bounds for reduced observables)**.** *Fix $R>0$ and let $\mathcal{O}$ be endpoint-admissible. Define $F_{\mathcal{O}}(x^-,x^+)=\mathcal{R}\mathcal{O}(x^-,x^+)$. Then on the ball $\{\|x^\pm\|_{\mathsf{H}}\le R\}$, $F_{\mathcal{O}}$ is separately d-Lipschitz in each endpoint, with*

$$\sup_{\|x^+\|_{\mathsf{H}}\le R}\operatorname{Lip}_d(x^-\mapsto F_{\mathcal{O}}(x^-,x^+))\le\max\Big\{\frac{2\|\mathcal{O}\|_\infty M_{\mathscr{J}}(t,R)}{\alpha},\ 2\|\mathcal{O}\|_\infty\Big\},\tag{120}$$

$$\sup_{\|x^-\|_{\mathsf{H}}\le R}\operatorname{Lip}_d(x^+\mapsto F_{\mathcal{O}}(x^-,x^+))\le\max\Big\{\frac{2\|\mathcal{O}\|_\infty M_{\mathscr{J}}(t,R)}{\alpha},\ 2\|\mathcal{O}\|_\infty\Big\}.\tag{121}$$

Proof. Fix $\|x^\pm\|_{\mathsf{H}}\le R$. By Lemma 10.6, $\|\nabla_{x^\pm}F_{\mathcal{O}}(x^-,x^+)\|_{\mathsf{H}}\le 2\|\mathcal{O}\|_\infty M_{\mathscr{J}}(t,R)$ on the ball. Thus $F_{\mathcal{O}}$ is $\|\cdot\|_{\mathsf{H}}$-Lipschitz in each endpoint separately with constant $2\|\mathcal{O}\|_\infty M_{\mathscr{J}}(t,R)$, and $|F_{\mathcal{O}}|\le\|\mathcal{O}\|_\infty$. Apply Lemma 10.4 to each endpoint map to obtain (120)–(121). ☐

10.4 Exponential clustering for time-separated slab observables

We now state the main "transport" consequence: slab observables supported in separated time blocks decorrelate exponentially, with the same rate ρ obtained for the boundary chain.

For $n\in\mathbb{Z}$, let $\mathcal{O}^{(n)}$ denote an observable depending only on the interior field $A^{(n)}$ on the slab $S^{(n)}_{t,L}$.

Theorem 10.8 (Clustering for one-slab observables in the stacked field)**.** *Assume (85) holds with constants (C,ρ) and assume $\pi(V)<\infty$. Let $\mathcal{O}$ and $\mathcal{P}$ be bounded one-slab observables on $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ such that their reductions $F_{\mathcal{O}}(x^-,x^+)=\mathcal{R}\mathcal{O}(x^-,x^+)$ and $F_{\mathcal{P}}(x^-,x^+)=\mathcal{R}\mathcal{P}(x^-,x^+)$ are endpoint-admissible.*

Define slab observables on the stacked field by

$$\mathcal{O}^{(0)}_{\mathrm{stack}}:=\mathcal{O}(A^{(0)}),\qquad\mathcal{P}^{(n)}_{\mathrm{stack}}:=\mathcal{P}(A^{(n)}).$$

Then, under $\mathbf{P}^{\mathrm{stack}}_{t,\mathrm{Reg}}$, for all $n\ge2$,

$$\big|\operatorname{Cov}(\mathcal{O}^{(0)}_{\mathrm{stack}},\mathcal{P}^{(n)}_{\mathrm{stack}})\big|\ \le\ \|\mathcal{O}\|_\infty\operatorname{Lip}^{(0)}_d(\mathbf{G}_{\mathcal{P}})\,C\,\rho^{n-1}\,\pi(V),\tag{122}$$

where $\mathbf{G}_{\mathcal{P}}:\mathsf{H}^2\to\mathbb{R}$ is given by

$$\mathbf{G}_{\mathcal{P}}(x_0,x_1):=F_{\mathcal{P}}(x_0,x_1),$$

and $\operatorname{Lip}^{(0)}_d(\mathbf{G}_{\mathcal{P}})$ is the reduced seminorm (97) (with $m=1$), i.e. $\operatorname{Lip}^{(0)}_d(\mathbf{G}_{\mathcal{P}})=\operatorname{Lip}_d(g_{\mathbf{G}_{\mathcal{P}}})$ for $g_{\mathbf{G}_{\mathcal{P}}}(x)=\mathbb{E}_x[F_{\mathcal{P}}(B_0,B_1)]$.

In particular, on the ball $\{\|x\|_{\mathsf{H}}\le R\}$ one has the explicit estimate

$$\big|\operatorname{Cov}(\mathcal{O}^{(0)}_{\mathrm{stack}},\mathcal{P}^{(n)}_{\mathrm{stack}})\big|\ \le\ \|\mathcal{O}\|_\infty\Big(\operatorname{Lip}_{d,1}(F_{\mathcal{P}})+\kappa_R\operatorname{Lip}_{d,2}(F_{\mathcal{P}})\Big)\,C\,\rho^{n-1}\,\pi(V),\qquad n\ge2,\tag{123}$$

where $\operatorname{Lip}_{d,1}$ (resp. $\operatorname{Lip}_{d,2}$) denotes the separate d-Lipschitz constant of $F_{\mathcal{P}}$ in its first (resp. second) endpoint on $\{\|x^\pm\|_{\mathsf{H}}\le R\}$, and

$$\kappa_R\ :=\ \sup_{\substack{x,x'\in\mathsf{H}\\ \|x\|_{\mathsf{H}},\|x'\|_{\mathsf{H}}\le R\\ x\ne x'}}\frac{\mathcal{W}_d(P(x,\cdot),P(x',\cdot))}{d(x,x')}\ \in[0,\infty).$$

Proof. By Lemma 10.3 with two nontrivial slabs,

$$\mathbb{E}\big[\mathcal{O}(A^{(0)})\,\mathcal{P}(A^{(n)})\big] = \mathbb{E}_{\mathbb{P}_\pi}\Big[F_{\mathcal{O}}(B_0,B_1)\,F_{\mathcal{P}}(B_n,B_{n+1})\Big],$$

and similarly $\mathbb{E}[\mathcal{O}(A^{(0)})] = \mathbb{E}_{\mathbb{P}_\pi}[F_{\mathcal{O}}(B_0,B_1)]$ and $\mathbb{E}[\mathcal{P}(A^{(n)})] = \mathbb{E}_{\mathbb{P}_\pi}[F_{\mathcal{P}}(B_n,B_{n+1})]$. Hence

$$\mathrm{Cov}\big(\mathcal{O}(A^{(0)}),\mathcal{P}(A^{(n)})\big) = \mathrm{Cov}_\pi\Big(F_{\mathcal{O}}(B_0,B_1),\ F_{\mathcal{P}}(B_n,B_{n+1})\Big). \tag{124}$$

Define the reduced single-site function $g_{\mathbf{G}_{\mathcal{P}}}$ (Definition 8.7 with $m=1$) by

$$g_{\mathbf{G}_{\mathcal{P}}}(x) \ = \ \mathbb{E}_x\big[F_{\mathcal{P}}(B_0,B_1)\big] = \int_{\mathsf{H}} F_{\mathcal{P}}(x,x')\,P(x,\,\mathrm{d}x').$$

For $n\geq 2$, condition on B_1 and use the Markov property:

$$\mathbb{E}_\pi\big[F_{\mathcal{P}}(B_n,B_{n+1})\,\big|\,B_1\big] = (P^{n-1}g_{\mathbf{G}_{\mathcal{P}}})(B_1), \qquad \mathbb{E}_\pi\big[F_{\mathcal{P}}(B_n,B_{n+1})\big] = \pi(g_{\mathbf{G}_{\mathcal{P}}}).$$

Therefore

$$\mathrm{Cov}_\pi\Big(F_{\mathcal{O}}(B_0,B_1),F_{\mathcal{P}}(B_n,B_{n+1})\Big) = \mathbb{E}_\pi\Big[F_{\mathcal{O}}(B_0,B_1)\Big((P^{n-1}g_{\mathbf{G}_{\mathcal{P}}})(B_1)-\pi(g_{\mathbf{G}_{\mathcal{P}}})\Big)\Big].$$

Using $|F_{\mathcal{O}}|\leq\|\mathcal{O}\|_\infty$ and Proposition 8.2 applied pointwise with $x=B_1$ and $f=g_{\mathbf{G}_{\mathcal{P}}}$, we obtain

$$\Big|(P^{n-1}g_{\mathbf{G}_{\mathcal{P}}})(B_1)-\pi(g_{\mathbf{G}_{\mathcal{P}}})\Big| \leq \mathrm{Lip}_d(g_{\mathbf{G}_{\mathcal{P}}})\,C\,\rho^{n-1}\,V(B_1).$$

Taking $\mathbb{E}_\pi[\cdot]$ and using stationarity gives

$$|\mathrm{Cov}(\mathcal{O}^{(0)}_{\mathrm{stack}},\mathcal{P}^{(n)}_{\mathrm{stack}})| \leq \|\mathcal{O}\|_\infty\,\mathrm{Lip}_d(g_{\mathbf{G}_{\mathcal{P}}})\,C\,\rho^{n-1}\,\pi(V),$$

which is (122) since $\mathrm{Lip}^{(0)}_d(\mathbf{G}_{\mathcal{P}}) = \mathrm{Lip}_d(g_{\mathbf{G}_{\mathcal{P}}})$.

For (123), fix $\|x\|_{\mathsf{H}},\|x'\|_{\mathsf{H}}\leq R$ and write

$$\begin{aligned}&|g_{\mathbf{G}_{\mathcal{P}}}(x)-g_{\mathbf{G}_{\mathcal{P}}}(x')|\\ &\leq \Big|\int F_{\mathcal{P}}(x,u)\,P(x,\,\mathrm{d}u)-\int F_{\mathcal{P}}(x',u)\,P(x,\,\mathrm{d}u)\Big| + \Big|\int F_{\mathcal{P}}(x',u)\,P(x,\,\mathrm{d}u)-\int F_{\mathcal{P}}(x',u)\,P(x',\,\mathrm{d}u)\Big|.\end{aligned}$$

The first term is bounded by $\mathrm{Lip}_{d,1}(F_{\mathcal{P}})\,d(x,x')$. For the second term, use Lemma 8.1 with the d-Lipschitz function $u\mapsto F_{\mathcal{P}}(x',u)$: it is d-Lipschitz in u with constant $\mathrm{Lip}_{d,2}(F_{\mathcal{P}})$, hence

$$\begin{aligned}\Big|\int F_{\mathcal{P}}(x',u)\,P(x,\,\mathrm{d}u)-\int F_{\mathcal{P}}(x',u)\,P(x',\,\mathrm{d}u)\Big| &\leq \mathrm{Lip}_{d,2}(F_{\mathcal{P}})\,\mathcal{W}_d(P(x,\cdot),P(x',\cdot))\\ &\leq \kappa_R\,\mathrm{Lip}_{d,2}(F_{\mathcal{P}})\,d(x,x').\end{aligned}$$

Thus, on the ball,

$$\mathrm{Lip}_d(g_{\mathbf{G}_{\mathcal{P}}}) \leq \mathrm{Lip}_{d,1}(F_{\mathcal{P}})+\kappa_R\,\mathrm{Lip}_{d,2}(F_{\mathcal{P}}).$$

and inserting this into (122) yields (123). □

Remark 10.9 (From step index to Euclidean time separation). Slab index separation n corresponds to Euclidean time separation $s=nt$ in the stacked field on $\mathbb{R}\times\mathbb{T}^3_L$. Thus (122) implies exponential clustering in Euclidean time with rate

$$\rho^{n-1} = \exp(-(n-1)|\log\rho|) \asymp \exp\Big(-\frac{|\log\rho|}{t}\,s\Big),$$

up to the discretisation shift by one slab.

10.5 Regulator-uniform transport: what must be uniform

The transport bounds above become regulator-uniform provided the following inputs are uniform in Reg:

(i) the boundary mixing constants C and ρ in (85);

(ii) the Lyapunov moment bound $\pi(V) \le M$ (e.g. via $\pi(V) \le K/(1-\lambda)$ from Lemma 8.3);

(iii) the response moment bound $M_{\mathscr{J}}(t, R)$ in (118) on the relevant radius R (Section 5);

(iv) the local kernel Lipschitz factor κ_R (equivalently, uniform control of $\mathcal{W}_d(P(x,\cdot), P(x',\cdot))$ on $\{\|x\|, \|x'\| \le R\}$), when using the explicit bound (123).

Under these uniformities, the right-hand side of (122) (and (123) when applicable) is uniform in Reg for fixed t and fixed observable norms.

Remark 10.10 (Scope and next step). Section 10 lifts boundary mixing to exponential clustering for a large class of slab-local observables in the stacked field construction. The remaining step needed to turn this into a full OS statement for the *four-dimensional* Euclidean gauge field is to relate the stacked field measure driven by the auxiliary boundary kernel $P = P_\tau$ to the target regulated Yang–Mills measure on $\mathbb{R} \times \mathbb{T}_L^3$ obtained from the slab-to-slab DLR/transfer-operator specification in the Euclidean time direction. This identification is addressed by the DLR/transfer-operator analysis in the next section.

11 DLR consistency and identification with the Euclidean slab measure

Sections 9–10 established two logically distinct ingredients:

(i) a *one-slab Euclidean* disintegration into endpoint data and conditional interior laws (Lemma 10.1);

(ii) an *abstract* mixing–to–time-axis exponential clustering calculus for a reversible stationary Markov kernel on the boundary space (Sections 8–9).

The purpose of the present section is to supply the missing wheel highlighted above: *define the Euclidean-time transfer kernel directly from the slab Gibbs weight by disintegration, prove exact concatenation (Chapman–Kolmogorov) in Euclidean time, and identify the stacked construction with the resulting Euclidean DLR measure.* No identification of an auxiliary sampling dynamics with Euclidean transfer is used or needed.

All arguments are at fixed regulator level Reg, so all configuration spaces are finite-dimensional (hence Polish).

11.1 Finite-volume Euclidean measures and slab factorisation

Fix $t > 0$ and $L > 0$. For $N \in \mathbb{N}$ define the N-slab cylinder

$$S_{t,L}^{(N)} := [0, Nt] \times \mathbb{T}_L^3,$$

and let $\mathsf{X}^{(N)}_{t,\mathrm{Reg}}$ be the regulated field configuration space on $S^{(N)}_{t,L}$ (with the same regulator Reg used on one slab). Let $\mathbf{m}^{(N)}_{t,\mathrm{Reg}}$ be the corresponding reference Gaussian measure and $S^{(N)}_{t,\mathrm{Reg}}(A)$ the regulated Euclidean action (including the fixed gauge-fixing terms used earlier). Define the unnormalised weight

$$\mathbf{w}^{(N)}_{t,\mathrm{Reg}}(\,\mathrm{d}A) \;\coloneqq\; e^{-S^{(N)}_{t,\mathrm{Reg}}(A)}\,\mathbf{m}^{(N)}_{t,\mathrm{Reg}}(\,\mathrm{d}A), \qquad A \in \mathsf{X}^{(N)}_{t,\mathrm{Reg}}. \tag{125}$$

The corresponding normalised Euclidean measure is

$$\mathbf{P}^{(N)}_{t,\mathrm{Reg}}(\,\mathrm{d}A) \;\coloneqq\; \frac{1}{\mathbf{Z}^{(N)}_{t,\mathrm{Reg}}}\,\mathbf{w}^{(N)}_{t,\mathrm{Reg}}(\,\mathrm{d}A), \qquad \mathbf{Z}^{(N)}_{t,\mathrm{Reg}} \coloneqq \int_{\mathsf{X}^{(N)}_{t,\mathrm{Reg}}} \mathbf{w}^{(N)}_{t,\mathrm{Reg}}(\,\mathrm{d}A). \tag{126}$$

For $k = 0, 1, \ldots, N$ let $\mathrm{Tr}_k : \mathsf{X}^{(N)}_{t,\mathrm{Reg}} \to \mathsf{H}$ denote the trace map at Euclidean time $x_0 = kt$, where $\mathsf{H} = \mathcal{B}_{L,\Lambda}$ is the regulated boundary space fixed in Section 7.1. Write $B_k \coloneqq \mathrm{Tr}_k(A)$.

We assume the *slab locality* (ensured by the regulator/gauge-fixing choices made in Sections 4–5) in the following explicit form.

Assumption 11.1 (Exact slab additivity and Gaussian Markov factorisation)**.** For each $N \geq 1$, the space $\mathsf{X}^{(N)}_{t,\mathrm{Reg}}$ admits an identification (a linear bijection) with the subspace of tuples $(A^{(0)}, \ldots, A^{(N-1)}) \in \prod_{n=0}^{N-1} \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ satisfying the matching constraints

$$\mathrm{Tr}_+(A^{(n)}) = \mathrm{Tr}_-(A^{(n+1)}), \qquad n = 0, \ldots, N-2,$$

where $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ is the one-slab space on $[0,t] \times \mathbb{T}^3_L$ and $\mathrm{Tr}_\pm$ are its endpoint traces. Under this identification:

(i) the action is additive,

$$S^{(N)}_{t,\mathrm{Reg}}(A^{(0)}, \ldots, A^{(N-1)}) = \sum_{n=0}^{N-1} S_{t,\mathrm{Reg}}(A^{(n)}); \tag{127}$$

(ii) the Gaussian reference measure is Markov in time and disintegrates exactly over the intermediate traces: for every bounded measurable Ψ on $\mathsf{X}^{(N)}_{t,\mathrm{Reg}}$,

$$\int \Psi\,\mathbf{m}^{(N)}_{t,\mathrm{Reg}}(\,\mathrm{d}A) = \int_{\mathsf{H}^{N+1}} \Big(\prod_{n=0}^{N-1} \mathbf{m}^{x_n,x_{n+1}}_{t,\mathrm{Reg}}(\,\mathrm{d}A^{(n)})\Big)\,\mu^{(N)}_{0,t,\mathrm{Reg}}(\,\mathrm{d}x_0, \ldots, \,\mathrm{d}x_N), \tag{128}$$

where $\mathbf{m}^{x_n,x_{n+1}}_{t,\mathrm{Reg}}$ is the one-slab Gaussian conditional law on the fibre $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}(x_n, x_{n+1})$, and $\mu^{(N)}_{0,t,\mathrm{Reg}}$ is the induced Gaussian law of the trace vector $(B_0, \ldots, B_N)$.

Assumption 11.1 is the exact statement that cutting the (regulated, gauge-fixed) free field at time slices kt produces a Gaussian Markov chain of traces and conditionally independent interiors, and that the interaction action is local in x_0. These are precisely the structural inputs needed for a DLR/transfer-operator construction in the Euclidean time direction.

Remark 11.2 (When Assumption 11.1 is expected to hold)*.* For lattice regularisations with a nearest-neighbour (in x_0) action and a time-local gauge fixing, the additivity (127) and the Gaussian Markov disintegration (128) are built into the construction: the free field is a Gaussian Markov random field in the time direction, and the interaction couples only adjacent time slices. For continuum regulators (e.g. Fourier cutoffs), the same structure holds provided the regulator respects time locality and the endpoint trace maps are compatible with the chosen Gaussian reference.

11.2 One-slab endpoint disintegration and the Euclidean transfer kernel

Recall from Lemma 10.1 the one-slab disintegration of the unnormalised weight $\mathbf{w}_{t,\mathrm{Reg}}(\mathrm{d}A) = e^{-S_{t,\mathrm{Reg}}(A)}\mathbf{m}_{t,\mathrm{Reg}}(\mathrm{d}A)$: there exists a finite endpoint measure $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$ on $\mathsf{H}\times\mathsf{H}$ and conditional interior laws $\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}$ such that

$$\int_{\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}} \Psi(A)\,\mathbf{w}_{t,\mathrm{Reg}}(\mathrm{d}A) = \int_{\mathsf{H}\times\mathsf{H}}\Big(\int_{\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}} \Psi(A)\,\mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}(\mathrm{d}A)\Big)\,\boldsymbol{\kappa}_{t,\mathrm{Reg}}(\mathrm{d}x^-,\,\mathrm{d}x^+) \tag{129}$$

for all bounded measurable Ψ.

Let $\boldsymbol{\kappa}^-_{t,\mathrm{Reg}}$ and $\boldsymbol{\kappa}^+_{t,\mathrm{Reg}}$ denote the first and second marginals of $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$.

Definition 11.3 (Euclidean trace marginal and transfer kernel)**.** Define the (normalised) one-time trace law

$$\pi^{\mathrm{tr}}_{t,\mathrm{Reg}}(\mathrm{d}x) \;:=\; \frac{1}{\boldsymbol{\kappa}^-_{t,\mathrm{Reg}}(\mathsf{H})}\,\boldsymbol{\kappa}^-_{t,\mathrm{Reg}}(\mathrm{d}x). \tag{130}$$

Let K_t be a version of the conditional endpoint kernel obtained by disintegrating $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$ with respect to $\boldsymbol{\kappa}^-_{t,\mathrm{Reg}}$:

$$\boldsymbol{\kappa}_{t,\mathrm{Reg}}(\mathrm{d}x,\,\mathrm{d}x') = \boldsymbol{\kappa}^-_{t,\mathrm{Reg}}(\mathrm{d}x)\,\mathsf{K}_t(x,\,\mathrm{d}x'), \qquad \mathsf{K}_t(x,\mathsf{H}) = 1 \;\text{ for } \boldsymbol{\kappa}^-_{t,\mathrm{Reg}}\text{-a.e. } x. \tag{131}$$

Remark 11.4. At the level of this paper's architecture, $\pi^{\mathrm{tr}}_{t,\mathrm{Reg}}$ is the *one-time trace marginal* induced by the Euclidean slab weight. Under the constructions in Sections 4–5, $\pi^{\mathrm{tr}}_{t,\mathrm{Reg}}$ coincides with the boundary law $\pi_{t,\mathrm{Reg}}$ introduced earlier in (75) (up to the normalisation convention for the one-slab weight). In what follows we write π for this common law.

11.3 Exact concatenation in Euclidean time

The key DL-style statement is that *Euclidean-time concatenation is exact.* Formally, gluing two slabs along a common interface and integrating out the interface trace produces composition of the transfer kernel, without any additional "perimeter factor."

Proposition 11.5 (Chapman–Kolmogorov for the Euclidean transfer kernel)**.** *Assume Assumption 11.1. Let K_t be the one-slab transfer kernel from Definition 11.3. Then the trace process at times $\{0,t,2t,\dots,Nt\}$ under the N-slab Euclidean measure $\mathbf{P}^{(N)}_{t,\mathrm{Reg}}$ is Markov with one-step transition K_t in the following sense:*

for each $N\ge 1$ and each $k\in\{0,\dots,N-1\}$,

$$\mathbf{P}^{(N)}_{t,\mathrm{Reg}}(B_{k+1}\in\cdot \mid B_0,\dots,B_k) = \mathsf{K}_t(B_k,\cdot) \qquad a.s. \tag{132}$$

Consequently, for all $m\le n$ the $n-m$ step transition satisfies the exact Chapman–Kolmogorov identity

$$\mathsf{K}_t^{n-m} = \underbrace{\mathsf{K}_t\circ\cdots\circ\mathsf{K}_t}_{n-m\ \ times}, \qquad i.e. \qquad \mathsf{K}_t^{n-m}(x,\,\mathrm{d}x') = \int \mathsf{K}_t^{n-k}(x,\,\mathrm{d}u)\,\mathsf{K}_t^{k-m}(u,\,\mathrm{d}x') \tag{133}$$

for any intermediate k with $m\le k\le n$.

Proof. Fix $N\ge1$ and $k\in\{0,\dots,N-1\}$. By Assumption 11.1 and the one-slab disintegration (129), conditional on the trace vector $(B_0,\dots,B_N)$ the slab interiors $(A^{(0)},\dots,A^{(N-1)})$ are independent with conditional laws $\mathbf{Q}^{B_n,B_{n+1}}_{t,\mathrm{Reg}}$. Moreover, the Radon–Nikodym factor $e^{-S_{t,\mathrm{Reg}}(A^{(n)})}$ is local to slab

n by (127). Therefore, conditioning on $(B_0, \dots, B_k)$ and integrating out the future interiors and future traces except B_{k+1} reduces exactly to the *one-slab* conditional distribution of the top trace given the bottom trace, which is $\mathsf{K}_t(B_k, \cdot)$ by (131). This yields (132).

The Chapman–Kolmogorov identity (133) is the standard semigroup property of iterated Markov transitions once (132) is established. □

11.4 Time-reversal symmetry, detailed balance, and the transfer operator

Lemma 11.6 (Time-reversal symmetry of the endpoint measure)**.** *Assume that the regulator and gauge-fixed slab weight is invariant under time reflection $x_0 \mapsto t - x_0$ (with the induced swap of endpoints $x^- \leftrightarrow x^+$). Then $\boldsymbol{\kappa}_{t,\mathrm{Reg}}$ is symmetric:*

$$\boldsymbol{\kappa}_{t,\mathrm{Reg}}(E \times F) = \boldsymbol{\kappa}_{t,\mathrm{Reg}}(F \times E) \qquad \text{for all Borel } E, F \subset \mathsf{H}. \tag{134}$$

In particular, $\boldsymbol{\kappa}^-_{t,\mathrm{Reg}} = \boldsymbol{\kappa}^+_{t,\mathrm{Reg}}$.

Proof. Let $\mathcal{R}$ denote the time-reflection map on one-slab fields: $(\mathcal{R}A)(x_0, x) := A(t - x_0, x)$ (in the chosen gauge-fixed representative). By the assumed symmetry, $\mathbf{w}_{t,\mathrm{Reg}}$ is invariant under $\mathcal{R}$ and the endpoint trace pair swaps:

$$(\mathrm{Tr}_-(\mathcal{R}A), \mathrm{Tr}_+(\mathcal{R}A)) = (\mathrm{Tr}_+(A), \mathrm{Tr}_-(A)).$$

Therefore, for Borel sets $E, F \subset \mathsf{H}$,

$$\begin{aligned} \boldsymbol{\kappa}_{t,\mathrm{Reg}}(E \times F) &= \mathbf{w}_{t,\mathrm{Reg}}(\{A : \ \mathrm{Tr}_-(A) \in E, \ \mathrm{Tr}_+(A) \in F\}) \\ &= \mathbf{w}_{t,\mathrm{Reg}}(\{A : \ \mathrm{Tr}_-(A) \in F, \ \mathrm{Tr}_+(A) \in E\}) = \boldsymbol{\kappa}_{t,\mathrm{Reg}}(F \times E). \end{aligned}$$

which is (134). Equality of marginals follows immediately. □

Proposition 11.7 (Invariance and detailed balance for K_t)**.** *Let π be the normalised trace law* (130)*. Under the symmetry assumption of Lemma 11.6, π is invariant for K_t and K_t is reversible with respect to π: for all bounded measurable f, g,*

$$\int f(x)\,(\mathsf{T}_t g)(x)\,\pi(\mathrm{d}x) = \int g(x)\,(\mathsf{T}_t f)(x)\,\pi(\mathrm{d}x), \qquad (\mathsf{T}_t f)(x) := \int f(x')\,\mathsf{K}_t(x, \mathrm{d}x'). \tag{135}$$

Proof. Write $\mathbf{c} := \boldsymbol{\kappa}^-_{t,\mathrm{Reg}}(\mathsf{H})$ so that $\pi = \mathbf{c}^{-1}\boldsymbol{\kappa}^-$. Invariance follows from (131) and $\boldsymbol{\kappa}^+ = \boldsymbol{\kappa}^-$:

$$\int \mathsf{K}_t(x, E)\,\pi(\mathrm{d}x) = \frac{1}{\mathbf{c}} \int \mathsf{K}_t(x, E)\,\boldsymbol{\kappa}^-(\mathrm{d}x) = \frac{1}{\mathbf{c}} \int \boldsymbol{\kappa}(\mathrm{d}x, E) = \frac{1}{\mathbf{c}}\boldsymbol{\kappa}^+(E) = \pi(E).$$

For detailed balance, combine (131) with symmetry (134):

$$\mathbf{c}\,\pi(\mathrm{d}x)\,\mathsf{K}_t(x, \mathrm{d}x') = \boldsymbol{\kappa}(\mathrm{d}x, \mathrm{d}x') = \boldsymbol{\kappa}(\mathrm{d}x', \mathrm{d}x) = \mathbf{c}\,\pi(\mathrm{d}x')\,\mathsf{K}_t(x', \mathrm{d}x).$$

Integrating $f(x)g(x')$ against both sides yields (135). □

Definition 11.8 (Euclidean transfer operator)**.** Define the Euclidean transfer operator T_t on $L^2(\pi)$ by

$$(\mathsf{T}_t f)(x) \ := \int_{\mathsf{H}} f(x')\,\mathsf{K}_t(x, \mathrm{d}x'). \tag{136}$$

By Proposition 11.7, T_t is a self-adjoint contraction on $L^2(\pi)$.

Proposition 11.9 (Covariance decay from an L^2 transfer contraction)**.** *Let $(B_n)_{n\in\mathbb{Z}}$ be the stationary bi-infinite boundary trace process with one-step kernel K_t and invariant law π (Proposition 11.7). Assume there exists $\rho \in (0,1)$ such that for all $f \in L^2(\pi)$ with $\pi(f) = 0$,*

$$\|\mathsf{T}_t^n f\|_{L^2(\pi)} \;\le\; \rho^n \,\|f\|_{L^2(\pi)}, \qquad n \ge 1, \tag{137}$$

where T_t is the transfer operator (136). Then for all $f, g \in L^2(\pi)$ and $n \ge 1$,

$$\bigl|\mathrm{Cov}_\pi\bigl(f(B_0), g(B_n)\bigr)\bigr| \;\le\; \rho^n \,\|f - \pi(f)\|_{L^2(\pi)}\, \|g - \pi(g)\|_{L^2(\pi)}.$$

Moreover, let F, G be bounded measurable functions on $\mathsf{H} \times \mathsf{H}$ and define the one-step reductions

$$f(x) \;\coloneqq\; \int F(x, x')\, \mathsf{K}_t(x, \mathrm{d}x'), \qquad g(x) \;\coloneqq\; \int G(x, x')\, \mathsf{K}_t(x, \mathrm{d}x').$$

Then for all $n \ge 2$,

$$\bigl|\mathrm{Cov}_\pi\bigl(F(B_0, B_1), G(B_n, B_{n+1})\bigr)\bigr| \;\le\; \rho^{n-1}\, \|f - \pi(f)\|_{L^2(\pi)}\, \|g - \pi(g)\|_{L^2(\pi)} \;\le\; 4\,\rho^{n-1}\, \|F\|_\infty \,\|G\|_\infty.$$

In particular, taking F and G to be the reduced endpoint functions of bounded interior-supported slab observables yields time-axis exponential clustering with rate parameter $|\log \rho|/t$.

Proof. For $f, g \in L^2(\pi)$ with $\pi(f) = 0$, stationarity and reversibility give

$$\mathrm{Cov}_\pi\bigl(f(B_0), g(B_n)\bigr) = \langle f, \mathsf{T}_t^n (g - \pi(g))\rangle_{L^2(\pi)},$$

so Cauchy–Schwarz and (137) yield the first bound.

For the two-step bound, note that by the Markov property, $\mathbb{E}[F(B_0, B_1) \mid B_0] = f(B_0)$ and $\mathbb{E}[G(B_n, B_{n+1}) \mid B_n] = g(B_n)$, hence

$$\mathrm{Cov}_\pi\bigl(F(B_0, B_1), G(B_n, B_{n+1})\bigr) = \mathrm{Cov}_\pi\bigl(f(B_0), g(B_n)\bigr),$$

and apply the first estimate with $n - 1$ steps. Finally, $\|f\|_\infty \le \|F\|_\infty$ and $\|g\|_\infty \le \|G\|_\infty$ imply $\|f - \pi(f)\|_2 \le 2\|F\|_\infty$ and $\|g - \pi(g)\|_2 \le 2\|G\|_\infty$. ☐

11.5 Exact DLR specification in the Euclidean time direction

We now express the finite-volume Euclidean measure $\mathbf{P}^{(N)}_{t,\mathrm{Reg}}$ in terms of: (i) the Euclidean transfer kernel K_t and (ii) the one-slab conditional interior laws $\mathbf{Q}^{x_n, x_{n+1}}_{t,\mathrm{Reg}}$.

Let $\nu^{(N)}_{t,\mathrm{Reg}}$ denote the law of the trace vector $(B_0, \ldots, B_N)$ under $\mathbf{P}^{(N)}_{t,\mathrm{Reg}}$.

Theorem 11.10 (Finite-volume DLR/transfer representation)**.** *Assume Assumption 11.1. Let K_t be the Euclidean transfer kernel from Definition 11.3. Write $\mathbf{Q}^{x,x'} \coloneqq \mathbf{Q}^{x,x'}_{t,\mathrm{Reg}}$.*

Then:

(i) *(*Trace-chain form*) The trace law $\nu^{(N)}_{t,\mathrm{Reg}}$ is Markov with transition K_t: there exists an initial law $\nu^{(N)}_{t,\mathrm{Reg},0}$ on H such that for all bounded measurable $\varphi : \mathsf{H}^{N+1} \to \mathbb{R}$,*

$$\mathbb{E}_{\mathbf{P}^{(N)}_{t,\mathrm{Reg}}}[\varphi(B_0, \ldots, B_N)] = \int_{\mathsf{H}} \nu^{(N)}_{t,\mathrm{Reg},0}(\mathrm{d}x_0) \int_{\mathsf{H}^N} \varphi(x_0, \ldots, x_N) \prod_{j=0}^{N-1} \mathsf{K}_t(x_j, \mathrm{d}x_{j+1}). \tag{138}$$

(ii) *(*DLR factorisation given traces*) Conditional on the trace vector* $(B_0, \dots, B_N)$*, the slab interiors* $A^{(0)}, \dots, A^{(N-1)}$ *are independent with conditional laws* $\mathbf{Q}^{B_n,B_{n+1}}$*. Consequently, whenever* $\mathcal{F}$ *factorises across slabs, i.e.* $\mathcal{F}(A) = \prod_{n=0}^{N-1} \mathcal{F}_n(A^{(n)})$*, with bounded measurable* $\mathcal{F}_n$*,*

$$\mathbb{E}_{\mathbf{P}^{(N)}_{t,\mathrm{Reg}}}[\mathcal{F}(A)] = \int_{\mathsf{H}} \nu^{(N)}_{t,\mathrm{Reg},0}(\,\mathrm{d}x_0) \int_{\mathsf{H}^N} \prod_{n=0}^{N-1} \Big(\int_{\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}} \mathcal{F}_n(A)\, \mathbf{Q}^{x_n,x_{n+1}}_{t,\mathrm{Reg}}(\,\mathrm{d}A) \Big) \prod_{j=0}^{N-1} \mathsf{K}_t(x_j,\, \mathrm{d}x_{j+1}). \tag{139}$$

Proof. Item (i) is the content of Proposition 11.5: (132) implies that $(B_0, \dots, B_N)$ is Markov with transition K_t. Taking $\nu^{(N)}_{t,\mathrm{Reg},0}$ to be the marginal law of B_0 under $\mathbf{P}^{(N)}_{t,\mathrm{Reg}}$ yields (138).

For (ii), Assumption 11.1 together with the one-slab disintegration (129) implies that, conditional on the trace vector, the interiors are conditionally independent with laws $\mathbf{Q}^{B_n,B_{n+1}}_{t,\mathrm{Reg}}$. Taking conditional expectation of $\prod_n \mathcal{F}_n(A^{(n)})$ given $(B_0, \dots, B_N)$ and then integrating over the trace law gives (139). □

Remark 11.11 (Equilibrium boundary condition). The initial law $\nu^{(N)}_{t,\mathrm{Reg},0}$ in (139) depends on the finite-volume boundary convention. For the *equilibrium* (stationary) time specification relevant for OS reconstruction and infinite-volume limits, one takes $\nu^{(N)}_{t,\mathrm{Reg},0} = \pi$. In that case the right-hand side of (139) depends only on $(\pi, \mathsf{K}_t, \mathbf{Q}^{\cdot,\cdot}_{t,\mathrm{Reg}})$ and is consistent under restriction to sub-windows.

11.6 Identification of the stacked field with the bi-infinite Euclidean DLR measure

We now define the bi-infinite Euclidean measure in the time direction as the (unique) DLR measure associated to the specification determined by $(\pi, \mathsf{K}_t, \mathbf{Q}^{\cdot,\cdot}_{t,\mathrm{Reg}})$.

Definition 11.12 (Bi-infinite Euclidean (time) Gibbs measure)**.** Let $\mathbb{P}^{\mathrm{tr}}_{\pi}$ be the stationary path measure of the Markov chain on $\mathsf{H}^{\mathbb{Z}}$ with invariant law π and transition kernel K_t. Define $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$ on

$$\Omega_{\mathrm{stack}} = \mathsf{H}^{\mathbb{Z}} \times \prod_{n\in\mathbb{Z}} \mathsf{X}^{(n)}_{t,\mathrm{Reg}}$$

by sampling $(B_n)_{n\in\mathbb{Z}} \sim \mathbb{P}^{\mathrm{tr}}_{\pi}$ and, conditional on (B_n), sampling independent slab interiors $A^{(n)} \sim \mathbf{Q}^{B_n,B_{n+1},(n)}_{t,\mathrm{Reg}}$.

Theorem 11.13 (DLR consistency and uniqueness)**.** *Assume Assumption 11.1. Then* $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$ *is DLR-consistent with the Euclidean time specification determined by* $(\pi, \mathsf{K}_t, \mathbf{Q}^{\cdot,\cdot}_{t,\mathrm{Reg}})$ *in the following sense:*

for any integers $m < n$ *and any bounded measurable functional* $\mathcal{F}$ *depending only on the slab interiors* $\{A^{(k)}\}_{k=m}^{n-1}$*, the conditional law of the window* $\{A^{(k)}\}_{k=m}^{n-1}$ *given* (B_m, B_n) *is obtained by: (i) sampling the intermediate boundaries* $(B_{m+1}, \dots, B_{n-1})$ *from the* K_t*-Markov bridge between* B_m *and* B_n*, and then (ii) sampling conditionally independent interiors with laws* $\mathbf{Q}^{B_k,B_{k+1},(k)}_{t,\mathrm{Reg}}$*.*

Moreover, $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$ *is the unique bi-infinite Gibbs measure on* Ω_{stack} *with this DLR specification.*

Proof. Under $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$, the boundary sequence $(B_n)_{n\in\mathbb{Z}}$ is a stationary Markov chain with kernel K_t. Hence, given endpoints (B_m, B_n), the intermediate boundaries follow the usual Markov bridge law induced by K_t, and conditional on the full boundary sequence the slab interiors are independent with laws $\mathbf{Q}^{B_k,B_{k+1}}_{t,\mathrm{Reg}}$. This is exactly the DLR window specification.

Uniqueness here is understood *relative to the fixed specification data* $(\pi, \mathsf{K}_t, \mathbf{Q})$: among probability measures on the bi-infinite time-axis whose one-time marginal equals π and whose finite-window conditional laws agree with the above DLR/bridge specification, the cylinder-event probabilities are determined uniquely. We do not claim uniqueness for an unconstrained Gibbs/DLR specification without fixing the equilibrium boundary condition; in general, multiple DLR states may exist. □

11.7 Consequences for transfer/OS time translation

The preceding identification is the conceptual repair: Euclidean time translation by one slab corresponds to the *Euclidean transfer operator* T_t of Definition 11.8. In particular:

(i) the path measure $\mathbb{P}_\pi^{\mathrm{tr}}$ is the Euclidean-time trace process induced by the regulated Yang–Mills slab specification;

(ii) reflection positivity of the trace process follows from detailed balance (Proposition 11.7);

(iii) the time-axis exponential clustering time clustering bounds proved abstractly for a reversible boundary kernel (Sections 8–9) are therefore Euclidean statements *provided they are applied to the Euclidean transfer kernel* K_t, not to any auxiliary sampling dynamics.

This completes the DLR/transfer-operator consistency step. In particular, the mixing/clustering bounds may be interpreted as genuine Euclidean-time correlation estimates for the regulated Yang–Mills slab construction.

12 From slab clustering to Schwinger-function clustering for gauge-invariant observables

This section formulates the exponential clustering consequences of Sections 10–11 directly in terms of *gauge-invariant Euclidean observables* (Wilson loops and bounded curvature functionals), and records the resulting *Schwinger-function* clustering bounds in the Euclidean time direction.

Conditionality. The sampler-to-observable transport mechanism recorded in this part of the manuscript (Section 10 and Subsections 12.2–12.5) belongs to the Gaussian-reference template: it uses Assumption 12.6 (uniform response moment bound) together with the density input Assumption 13.11 to obtain Theorem 12.11 and its corollaries. In the Wilson KP corridor of Theorem 2.14 we do *not* invoke this transport assumption; instead, exponential time-axis clustering is deduced directly from the transfer-operator L^2 gap in Theorem 2.14(ii) (equivalently Corollary 2.25) via Proposition 11.9, and it persists under $L \to \infty$ within the L-uniform corridor of Theorem I.1. **Two routes.** Route A (template sampler ⇒ slab observables) uses Assumptions 12.6 and 13.11. Route B (Wilson/KP corridor) bypasses transport entirely and yields clustering for lattice observables in $\mathfrak{Obs}_{t,\delta}^{\mathrm{lat}}$ directly from the transfer-operator L^2 contraction (Theorem 2.14(ii)–(iii) and Proposition 11.9).

All statements are at fixed regulator level Reg and fixed spatial torus $\mathbb{T}_L^3$. The underlying Euclidean-time infinite-volume measure in the time direction is the DLR/stacked measure $\mathbf{P}_{t,\mathrm{Reg}}^{\infty}$ from Definition 11.12, built from the one-slab conditionals $\mathbf{Q}_{t,\mathrm{Reg}}^{x^-,x^+}$ and the Euclidean transfer kernel K_t (Section 11). Expectation with respect to $\mathbf{P}_{t,\mathrm{Reg}}^{\infty}$ is denoted $\mathbb{E}_\infty[\cdot]$.

12.1 Regulated gauge fields and gauge transformations

Let G be a fixed compact Lie group and $\mathfrak{g}$ its Lie algebra, equipped with an Ad-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. At regulator level Reg, the one-slab configuration space $\mathsf{X}_{t,\mathrm{Reg}}^{\mathrm{temp}}$ (Section 4) is a finite-dimensional real vector space of $\mathfrak{g}$-valued one-forms on $S_{t,L} = [0,t] \times \mathbb{T}^3_L$ (e.g. Fourier-truncated smooth forms, possibly after a fixed gauge choice).

Let $\mathcal{G}_{t,\mathrm{Reg}}$ be the corresponding finite-dimensional gauge group of G-valued maps on $S_{t,L}$ compatible with the regulator (e.g. Fourier-truncated smooth maps), acting on $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ by

$$A \mapsto A^g \;\coloneqq\; \mathrm{Ad}_{g^{-1}} A + g^{-1}\, \mathrm{d}g, \qquad g \in \mathcal{G}_{t,\mathrm{Reg}}. \tag{140}$$

(Any fixed gauge-fixing used to define the slab weight is understood as part of the definition of the measure; the notion of gauge-invariant *observables* below is defined with respect to (140) and does not depend on gauge fixing.)

Definition 12.1 (Gauge-invariant slab observable)**.** A measurable function $\mathcal{O} : \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} \to \mathbb{R}$ is called *gauge-invariant* if

$$\mathcal{O}(A^g) = \mathcal{O}(A) \qquad \text{for all } A \in \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}},\ g \in \mathcal{G}_{t,\mathrm{Reg}}. \tag{141}$$

12.2 Admissible gauge-invariant cylinder observables

The mixing-to-clustering mechanism of Sections 10–8 requires *bounded* slab observables whose reduced expectations are *endpoint-admissible* (Definition 10.5), with quantitative control provided by the response moment bounds (Section 5).

We therefore restrict attention to gauge-invariant observables supported strictly in the slab interior, so that boundary perturbations are mediated by the quasi-local response kernels of Section 4.

For $\delta \in (0, t/2)$ define the interior sub-slab

$$S^{(\delta)}_{t,L} \;\coloneqq\; [\delta, t-\delta] \times \mathbb{T}^3_L.$$

Definition 12.2 (Interior-supported cylinder observable)**.** A measurable $\mathcal{O} : \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} \to \mathbb{R}$ is called *δ-interior-supported* if $\mathcal{O}(A)$ depends only on the restriction $A|_{S^{(\delta)}_{t,L}}$.

Definition 12.3 (Gauge-invariant admissible class $\mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$)**.** Fix $\delta \in (0, t/2)$. Let $\mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$ be the class of slab observables $\mathcal{O} : \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} \to \mathbb{R}$ such that:

(i) $\mathcal{O}$ is bounded: $\|\mathcal{O}\|_\infty < \infty$;

(ii) $\mathcal{O}$ is δ-interior-supported (Definition 12.2);

(iii) $\mathcal{O}$ is gauge-invariant (Definition 12.1).

12.2.1 Wilson loops

Let $\gamma : [0,1] \to S^{(\delta)}_{t,L}$ be a piecewise C^1 closed loop. For $A \in \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$, define the parallel transport $U_{\gamma,A} : [0,1] \to G$ as the solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}s} U_{\gamma,A}(s) \;=\; -\,A(\dot\gamma(s))\, U_{\gamma,A}(s), \qquad U_{\gamma,A}(0) = \mathbf{1}. \tag{142}$$

Since A is smooth at fixed regulator level and γ is piecewise C^1, the ODE has a unique global solution. Define the holonomy $\mathrm{Hol}_\gamma(A) \coloneqq U_{\gamma,A}(1) \in G$.

Let $\rho : G \to U(d_\rho)$ be a finite-dimensional unitary representation and let $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$ be its character. Define the normalised Wilson loop observable

$$\mathcal{W}_{\gamma,\rho}(A) \;\coloneqq\; \frac{1}{d_\rho}\, \chi_\rho(\mathrm{Hol}_\gamma(A)). \tag{143}$$

Lemma 12.4 (Wilson loops are bounded, gauge-invariant, and interior-supported)**.** *For any $\delta \in (0, t/2)$ and any loop $\gamma \subset S_{t,L}^{(\delta)}$, the observable $\mathcal{W}_{\gamma,\rho}$ lies in $\mathfrak{Obs}_{t,\delta}^{\text{temp}}$. Moreover,*

$$\|\mathcal{W}_{\gamma,\rho}\|_\infty \;\leq\; 1. \tag{144}$$

Proof. Boundedness. Since $\rho(g)$ is unitary, all eigenvalues of $\rho(g)$ lie on the unit circle, hence $|\chi_\rho(g)| \leq d_\rho$. Therefore $|\mathcal{W}_{\gamma,\rho}(A)| \leq 1$ for all A, which is (144).

Interior support. If $\gamma \subset S_{t,L}^{(\delta)}$, then $\mathrm{Hol}_\gamma(A)$ depends only on A along γ, hence only on $A|_{S_{t,L}^{(\delta)}}$.

Gauge invariance. Under (140), holonomy transforms by conjugation: holonomy transforms by conjugation along the basepoint:

$$\mathrm{Hol}_\gamma(A^g) = g(\gamma(0))^{-1}\,\mathrm{Hol}_\gamma(A)\,g(\gamma(0)),$$

for closed loops. Characters are conjugation invariant, so $\chi_\rho(\mathrm{Hol}_\gamma(A^g)) = \chi_\rho(\mathrm{Hol}_\gamma(A))$. Thus (141) holds. □

12.2.2 Bounded curvature-energy observables

Let $F(A)$ denote the curvature two-form of A, defined (at fixed regulator level) by the usual expression $F(A) = \mathrm{d}A + \frac{1}{2}[A \wedge A]$. Let $\phi \in C_c^\infty(S_{t,L}^{(\delta)})$ be a real-valued test function supported in $S_{t,L}^{(\delta)}$. Define the (gauge-invariant) smeared Yang–Mills energy density

$$\mathcal{E}_\phi(A) \;\coloneqq\; \int_{S_{t,L}} \phi(x)\,\langle F_{\mu\nu}(A)(x),\, F_{\mu\nu}(A)(x)\rangle_{\mathfrak{g}}\,\mathrm{d}x, \tag{145}$$

with the usual summation convention in Euclidean coordinates and $\mathrm{d}x$ the volume form on $S_{t,L}$.

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be bounded measurable (typically bounded Lipschitz). Define the bounded curvature observable

$$\mathcal{O}_{\phi,\Psi}(A) \;\coloneqq\; \Psi(\mathcal{E}_\phi(A)). \tag{146}$$

Lemma 12.5 (Bounded curvature observables lie in $\mathfrak{Obs}_{t,\delta}^{\text{temp}}$)**.** *If $\phi \in C_c^\infty(S_{t,L}^{(\delta)})$ and Ψ is bounded, then $\mathcal{O}_{\phi,\Psi} \in \mathfrak{Obs}_{t,\delta}^{\text{temp}}$ and $\|\mathcal{O}_{\phi,\Psi}\|_\infty \leq \|\Psi\|_{L^\infty(\mathbb{R})}$.*

Proof. Boundedness is immediate from (146). Interior support holds because ϕ is supported in $S_{t,L}^{(\delta)}$. Gauge invariance holds because $F(A^g) = \mathrm{Ad}_{g^{-1}}F(A)$ and the inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is Ad-invariant, so the integrand in (145) is invariant. □

12.3 Endpoint admissibility and response control for $\mathfrak{Obs}_{t,\delta}^{\text{temp}}$

We now show that observables in $\mathfrak{Obs}_{t,\delta}^{\text{temp}}$ fit the admissibility framework of Section 10. The key quantitative input is the response identity and its uniform moment bounds from Sections 4 and 5, restated abstractly in Lemma 10.6.

Assumption 12.6 (Uniform response moment bound)**.** Fix $\delta \in (0, t/2)$. There exists a constant $M_{\mathscr{J}}(t,\delta) < \infty$ such that for the response density $\mathscr{J}_{t,\mathrm{Reg}}(x^-, x^+; A)$ appearing in Lemma 10.6,

$$\sup_{x^-,x^+\in \mathsf{H}} \mathbb{E}_{\mathbf{Q}_{t,\mathrm{Reg}}^{x^-,x^+}}\big[\|\mathscr{J}_{t,\mathrm{Reg}}(x^-,x^+;A)\|_{\mathsf{H}}\big] \;\leq\; M_{\mathscr{J}}(t,\delta). \tag{147}$$

Assumption 12.6 is exactly the output of Section 5 specialised to observables supported away from the time boundaries by δ.

Remark 12.7 (When Assumption 12.6 is expected to hold)*.* The bound (147) is a quantitative "quasi-local response" statement: interior observables supported a distance δ away from the time boundaries have endpoint derivatives whose H-norm moments are uniformly bounded. For standard local regulators, such bounds are typically obtained from locality of the conditional law together with uniform (in Reg) moment/exponential-moment estimates on the bulk fields on unit blocks, plus decay of response kernels away from the support of the observable. The paper treats (147) as a clean modular hypothesis because its proof is the place where one pays the full renormalisation/regularisation cost in a concrete Yang–Mills construction.

Remark 12.8 (On endpoint-response moment bounds in the Wilson corridor)*.* The abstract transport mechanism of Sections 10 and 12 is formulated for the optional Euclidean/Gaussian template and uses the endpoint-response moment input of Assumption 12.6. In the concrete Wilson lattice KP corridor, we do *not* use this template route: exponential Euclidean-time clustering and the associated transfer-operator consequences are obtained directly from the KP/cluster expansion for local gauge-invariant observables (Appendix H) and persist under $L \to \infty$ within the L-uniform corridor of Theorem I.1.

Proposition 12.9 (Endpoint admissibility and global d-Lipschitz bound)**.** *Fix $\delta \in (0, t/2)$ and assume Assumption 12.6. Let $\mathcal{O} \in \mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$. Define its reduced endpoint function*

$$F_{\mathcal{O}}(x^-, x^+) \;:=\; \int_{\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}} \mathcal{O}(A)\, \mathbf{Q}^{x^-,x^+}_{t,\mathrm{Reg}}(\,\mathrm{d}A).$$

Then:

(i) *$\mathcal{O}$ is endpoint-admissible (Definition 10.5) and the response identity (116) holds for all $x^{\pm} \in \mathsf{H}$ and all $h \in \mathsf{H}$;*

(ii) *for each fixed $x^- \in \mathsf{H}$, the map $x^+ \mapsto F_{\mathcal{O}}(x^-, x^+)$ is globally d-Lipschitz with*

$$\mathrm{Lip}_d\big(x^+ \mapsto F_{\mathcal{O}}(x^-, x^+)\big) \;\le\; \max\Big\{\frac{2\|\mathcal{O}\|_\infty\, M_{\mathscr{J}}(t,\delta)}{\alpha},\; 2\|\mathcal{O}\|_\infty\Big\}. \tag{148}$$

Proof. Item (i) is Lemma 10.6 applied with the uniform bound (147), which removes any restriction to bounded endpoint balls.

Item (ii) follows by repeating the proof of Proposition 10.7 with $M_{\mathscr{J}}(t,\delta)$ in place of $M_{\mathscr{J}}(t,R)$. □

12.4 Schwinger functions in discrete Euclidean time

Under $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$, the bi-infinite field consists of slab interiors $(A^{(n)})_{n\in\mathbb{Z}}$. For a one-slab observable $\mathcal{O} : \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} \to \mathbb{R}$, define its translate to slab n by

$$\mathcal{O}^{[n]} \;:=\; \mathcal{O}(A^{(n)}).$$

Definition 12.10 (Two-point Schwinger function and connected part)**.** For bounded slab observables $\mathcal{O}, \mathcal{P}$, define the two-point Schwinger function at discrete time separation $n \in \mathbb{Z}$ by

$$S_{\mathcal{O},\mathcal{P}}(n) \;:=\; \mathbb{E}_\infty\big[\mathcal{O}^{[0]}\, \mathcal{P}^{[n]}\big]. \tag{149}$$

Define the connected (truncated) two-point function by

$$S^{\mathrm{conn}}_{\mathcal{O},\mathcal{P}}(n) \;:=\; S_{\mathcal{O},\mathcal{P}}(n) - \mathbb{E}_\infty[\mathcal{O}^{[0]}]\, \mathbb{E}_\infty[\mathcal{P}^{[0]}] = \mathrm{Cov}_\infty\big(\mathcal{O}^{[0]}, \mathcal{P}^{[n]}\big). \tag{150}$$

12.5 Exponential clustering for gauge-invariant Schwinger functions

Let (C, ρ) denote the geometric mixing constants for the *Euclidean transfer chain* with kernel K_t as used in Sections 8–9, and let $\pi(V) < \infty$ be the Lyapunov moment bound (Lemma 8.3). These are entries in the constants ledger.

Theorem 12.11 (Exponential clustering of gauge-invariant two-point Schwinger functions)**.** *Fix $\delta \in (0, t/2)$ and assume Assumption 12.6. Let $\mathcal{O}, \mathcal{P} \in \mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$. Then for all integers $n \geq 2$,*

$$|S^{\mathrm{conn}}_{\mathcal{O},\mathcal{P}}(n)| \;\leq\; \|\mathcal{O}\|_\infty \; \max\Big\{\frac{2\|\mathcal{P}\|_\infty \, M_{\mathscr{J}}(t,\delta)}{\alpha},\; 2\|\mathcal{P}\|_\infty\Big\}\; C\,\rho^{n-1}\,\pi(V). \tag{151}$$

Equivalently, writing the Euclidean time separation as $s = nt$,

$$|S^{\mathrm{conn}}_{\mathcal{O},\mathcal{P}}(n)| \;\leq\; \mathsf{C}_{\mathcal{O},\mathcal{P}}(t,\delta)\,\exp(-m_*\,s), \qquad m_* \;\coloneqq\; \frac{|\log\rho|}{t}, \tag{152}$$

with

$$\mathsf{C}_{\mathcal{O},\mathcal{P}}(t,\delta) \;\coloneqq\; \|\mathcal{O}\|_\infty \; \max\Big\{\frac{2\|\mathcal{P}\|_\infty \, M_{\mathscr{J}}(t,\delta)}{\alpha},\; 2\|\mathcal{P}\|_\infty\Big\}\; C\,\rho^{-1}\,\pi(V). \tag{153}$$

Proof. By Definition 11.12 and Lemma 10.3, the stacked Euclidean field satisfies the reduction identity with boundary chain (B_n) having stationary law π and transition kernel K_t: for $n \geq 2$,

$$\mathbb{E}_\infty\big[\mathcal{O}^{[0]}\,\mathcal{P}^{[n]}\big] = \mathbb{E}_{\mathbb{P}^{\mathrm{tr}}_\pi}\big[F_{\mathcal{O}}(B_0,B_1)\,F_{\mathcal{P}}(B_n,B_{n+1})\big],$$

where

$$F_{\mathcal{O}}(x_0,x_1) \;\coloneqq\; \int \mathcal{O}(A)\,\mathbf{Q}^{x_0,x_1}_{t,\mathrm{Reg}}(\,\mathrm{d}A), \qquad F_{\mathcal{P}}(x_0,x_1) \;\coloneqq\; \int \mathcal{P}(A)\,\mathbf{Q}^{x_0,x_1}_{t,\mathrm{Reg}}(\,\mathrm{d}A).$$

The same reduction holds for the one-point expectations, hence

$$S^{\mathrm{conn}}_{\mathcal{O},\mathcal{P}}(n) = \mathrm{Cov}_{\mathbb{P}^{\mathrm{tr}}_\pi}\Big(F_{\mathcal{O}}(B_0,B_1),\; F_{\mathcal{P}}(B_n,B_{n+1})\Big).$$

Now apply Theorem 10.8 (transported window-mixing) to the Euclidean transfer chain. To verify the Lipschitz input for the "future" observable, note that Proposition 12.9 implies that, for each fixed first argument x_0, the map $x_1 \mapsto F_{\mathcal{P}}(x_0,x_1)$ is globally d-Lipschitz with bound (148). Therefore the one-sided window seminorm in the second coordinate satisfies

$$\mathrm{Lip}^{(0)}_d(\mathbf{G}_{\mathcal{P}}) \leq \max\Big\{\frac{2\|\mathcal{P}\|_\infty \, M_{\mathscr{J}}(t,\delta)}{\alpha},\; 2\|\mathcal{P}\|_\infty\Big\}, \qquad \mathbf{G}_{\mathcal{P}}(x_0,x_1) = F_{\mathcal{P}}(x_0,x_1).$$

Also $\|F_{\mathcal{O}}\|_\infty \leq \|\mathcal{O}\|_\infty$. Substituting into (123) yields (151).

Finally, (152) follows from $\rho^{n-1} = \rho^{-1}\exp(-(nt)\,|\log\rho|/t) = \rho^{-1}\exp(-m_* s)$ and the definition (153). □

Corollary 12.12 (Examples: Wilson loops and bounded curvature observables)**.** *Fix $\delta \in (0, t/2)$ and assume Assumption 12.6. Let $\gamma_0, \gamma_1 \subset S^{(\delta)}_{t,L}$ be loops and let ρ_0, ρ_1 be unitary representations of G. Then the connected Schwinger function of the corresponding Wilson loops satisfies, for $n \geq 2$,*

$$\big|\mathrm{Cov}_\infty(\mathcal{W}^{[0]}_{\gamma_0,\rho_0}, \mathcal{W}^{[n]}_{\gamma_1,\rho_1})\big| \;\leq\; \max\Big\{\frac{2\,M_{\mathscr{J}}(t,\delta)}{\alpha},\; 2\Big\}\; C\,\rho^{n-1}\,\pi(V),$$

using $\|\mathcal{W}_{\gamma,\rho}\|_\infty \leq 1$ from Lemma 12.4.

Likewise, for $\phi_0, \phi_1 \in C^\infty_c(S^{(\delta)}_{t,L})$ and bounded Ψ_0, Ψ_1, the connected Schwinger function of $\mathcal{O}_{\phi_i,\Psi_i}$ satisfies (151) with $\|\mathcal{O}_{\phi_i,\Psi_i}\|_\infty \leq \|\Psi_i\|_\infty$.

12.6 Block clustering for multi-time gauge-invariant observables

For completeness, we record the multi-time extension in a form that follows directly from the window-correlation bound (Proposition 8.8) combined with the reduction argument.

Let $\mathcal{O}_0,\dots,\mathcal{O}_m \in \mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$ and $\mathcal{P}_0,\dots,\mathcal{P}_m \in \mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$. Define block observables supported on $m+1$ consecutive slabs by

$$\mathcal{F} \;:=\; \prod_{j=0}^{m} \mathcal{O}_j^{[j]}, \qquad \mathcal{G}_n \;:=\; \prod_{j=0}^{m} \mathcal{P}_j^{[n+j]}, \qquad n \ge m+2.$$

Then, repeating the reduction to the Euclidean transfer chain and applying Proposition 8.8 (with window length $m+1$), one obtains an exponential covariance bound

$$\bigl|\mathrm{Cov}_\infty(\mathcal{F},\mathcal{G}_n)\bigr| \;\le\; \Bigl(\prod_{j=0}^{m} \|\mathcal{O}_j\|_\infty\Bigr)\, \mathsf{L}_{\mathcal{P}}(t,\delta)\, C\, \rho^{\,n-(m+1)}\, \pi(V), \tag{154}$$

where $\mathsf{L}_{\mathcal{P}}(t,\delta)$ is an explicit one-sided d-Lipschitz constant. Writing $M_{\mathcal{P}} := \max_{0\le j\le m} \|\mathcal{P}_j\|_\infty$, Proposition 12.9 yields

$$\mathsf{L}_{\mathcal{P}}(t,\delta) \;\le\; \max\Bigl\{\frac{2}{\alpha}\, M_{\mathcal{P}}\, M_{\mathscr{J}}(t,\delta),\; 2M_{\mathcal{P}}\Bigr\}. \tag{155}$$

13 OS reconstruction in Euclidean time and the transfer-operator gap

This section constructs the Osterwalder–Schrader (OS) Hilbert space for the *regulated Euclidean theory in the Euclidean-time direction* (i.e. the bi-infinite DLR/stacked measure of Section 11), identifies the Euclidean time-translation by one slab with the transfer operator T_t, and proves a spectral consequence: *exponential time-clustering implies a spectral gap above the vacuum* for T_t on the orthogonal complement of constants (at fixed regulator).

All constructions are carried out at fixed regulator level Reg and fixed $t>0$ and $L>0$.

13.1 Path space, time reflection, and the positive-time algebra

Recall the bi-infinite DLR/stacked measure $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$ of Definition 11.12 on the space

$$\Omega_{\mathrm{stack}} \;=\; \mathsf{H}^{\mathbb{Z}} \times \prod_{n\in\mathbb{Z}} \mathsf{X}^{(n)}_{t,\mathrm{Reg}},$$

with boundary trace process $(B_n)_{n\in\mathbb{Z}}$ and slab interiors $(A^{(n)})_{n\in\mathbb{Z}}$. Here $A^{(n)}$ is the interior field on the slab $S^{(n)}_{t,L} = [nt,(n+1)t]\times\mathbb{T}^3_L$, identified with an element of the canonical one-slab space $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ by translation to $[0,t]\times\mathbb{T}^3_L$.

Let $\mathscr{F}$ denote the product Borel σ-algebra on Ω_{stack}. Let $\mathscr{F}_+$ be the σ-algebra generated by the coordinates

$$\{\, B_n,\; A^{(n)} \;:\; n\ge 0\,\},$$

and let $\mathscr{F}_-$ be the σ-algebra generated by

$$\{\, B_n \;:\; n\le 0\,\} \;\cup\; \{\, A^{(n)} \;:\; n\le -1\,\}.$$

Thus $\mathscr{F}_+$ and $\mathscr{F}_-$ intersect on the “time-0” boundary variable B_0.

Fix once and for all the one-slab time-reflection map

$$\mathcal{R} : \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}} \to \mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}, \qquad (\mathcal{R}A)(x_0, x) := A(t - x_0, x), \tag{156}$$

which swaps the traces $\mathrm{Tr}_-(\mathcal{R}A) = \mathrm{Tr}_+(A)$ and $\mathrm{Tr}_+(\mathcal{R}A) = \mathrm{Tr}_-(A)$.

Definition 13.1 (Full time reflection on Ω_{stack})**.** Define $\Theta : \Omega_{\mathrm{stack}} \to \Omega_{\mathrm{stack}}$ by

$$(\Theta B)_n := B_{-n}, \qquad (\Theta A)^{(n)} := \mathcal{R}\big(A^{(-n-1)}\big), \qquad n \in \mathbb{Z}. \tag{157}$$

Definition 13.2 (Positive-time cylinder algebra)**.** Let $\mathcal{A}^{\mathrm{full}}_+$ be the set of bounded cylinder functions F on Ω_{stack} that are $\mathscr{F}_+$-measurable, i.e. depend only on finitely many coordinates

$$(B_0, \dots, B_{m+1}, A^{(0)}, \dots, A^{(m)})$$

for some $m \ge 0$.

13.2 Reflection positivity for the full Euclidean measure

The OS inner product is defined from the reflection pairing $F \mapsto \Theta F$. The key identity is that $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$ is reflection-positive on $\mathcal{A}^{\mathrm{full}}_+$, and the corresponding pairing can be represented as an $L^2(\pi)$ inner product on the boundary trace space.

Throughout this subsection, π and K_t are the normalised trace law and transfer kernel from Definitions 2.4–2.7, and T_t is the associated transfer operator on $L^2(\pi)$ (Definition 11.8). We write $\mathbb{E}_\infty[\cdot]$ for expectation under $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$.

Definition 13.3 (OS conditional expectation map)**.** For $F \in \mathcal{A}^{\mathrm{full}}_+$ define

$$\Phi_{\mathrm{full}}(F)(x) \;:=\; \mathbb{E}_\infty\big[F \,\big|\, B_0 = x\big], \qquad x \in \mathsf{H}. \tag{158}$$

Lemma 13.4 (Well-definedness and L^2-integrability)**.** *For each $F \in \mathcal{A}^{\mathrm{full}}_+$, the function $\Phi_{\mathrm{full}}(F)$ is bounded and belongs to $L^2(\pi)$. Moreover,*

$$\|\Phi_{\mathrm{full}}(F)\|_{L^\infty(\mathsf{H})} \;\le\; \|F\|_{L^\infty(\Omega_{\mathrm{stack}})}. \tag{159}$$

Proof. Since F is bounded, conditional expectation preserves the essential supremum, yielding (159). Thus $\Phi_{\mathrm{full}}(F) \in L^\infty(\mathsf{H})$, hence $\Phi_{\mathrm{full}}(F) \in L^2(\pi)$ because π is a probability measure. $\square$

Lemma 13.5 (Θ-invariance)**.** *Assume the one-slab unnormalised weight is invariant under the slab reflection $\mathcal{R}$ (156). Then the stacked measure $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$ is invariant under Θ:*

$$\Theta_\# \mathbf{P}^{\infty}_{t,\mathrm{Reg}} = \mathbf{P}^{\infty}_{t,\mathrm{Reg}}.$$

Proof. By Lemma 11.6 and Proposition 11.7, the boundary chain with kernel K_t and invariant law π is stationary and reversible, hence invariant under time reflection $n \mapsto -n$. Conditional on the boundary sequence, slab interiors are independent with one-slab conditional laws $\mathbf{Q}^{x_n, x_{n+1}}_{t,\mathrm{Reg}}$; invariance of the one-slab weight under $\mathcal{R}$ implies that $\mathbf{Q}^{x,x'}_{t,\mathrm{Reg}}$ is carried to $\mathbf{Q}^{x',x}_{t,\mathrm{Reg}}$ under $\mathcal{R}$ (endpoint swap). These two facts match exactly the definition (157), so the full stacked law is Θ-invariant. $\square$

Theorem 13.6 (Full reflection-positivity identity)**.** *Assume the one-slab unnormalised weight is invariant under the slab reflection $\mathcal{R}$ (156). Then for all $F, G \in \mathcal{A}^{\mathrm{full}}_+$,*

$$\mathbb{E}_\infty\big[\overline{F}\,(\Theta G)\big] \;=\; \langle \Phi_{\mathrm{full}}(F),\, \Phi_{\mathrm{full}}(G)\rangle_{L^2(\pi)}. \tag{160}$$

In particular,

$$\mathbb{E}_\infty\big[\overline{F}\,(\Theta F)\big] \;\ge\; 0, \qquad F \in \mathcal{A}^{\mathrm{full}}_+. \tag{161}$$

Proof. Step 1: reduction to a generating algebra. By a standard monotone-class argument, it suffices to prove (160) for bounded cylinder functions of the form

$$F(\omega) = \prod_{j=0}^{m} \mathcal{O}_j(A^{(j)})\,\varphi(B_{m+1}), \qquad G(\omega) = \prod_{j=0}^{n} \mathcal{P}_j(A^{(j)})\,\psi(B_{n+1}), \tag{162}$$

where each $\mathcal{O}_j, \mathcal{P}_j$ is bounded measurable on $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$ and φ, ψ are bounded measurable on H. Such functions generate $\mathscr{F}_+$.

Step 2: compute $\Phi_{\mathrm{full}}(F)$ and $\Phi_{\mathrm{full}}(G)$. Fix F as in (162). Under $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$, conditional on $B_0 = x_0$, the future boundary chain $(B_1, B_2, \dots)$ evolves by K_t, and conditional on successive boundary pairs (B_j, B_{j+1}), the slab interiors $A^{(j)}$ are independent with laws $\mathbf{Q}^{B_j,B_{j+1}}_{t,\mathrm{Reg}}$. Therefore, iterating conditional expectations yields

$$\Phi_{\mathrm{full}}(F)(x_0) = \int_{\mathsf{H}^{m+1}} \varphi(x_{m+1}) \prod_{j=0}^{m} F_{\mathcal{O}_j}(x_j, x_{j+1}) \prod_{j=0}^{m} \mathsf{K}_t(x_j, \mathrm{d}x_{j+1}), \qquad (x_0 \text{ fixed}). \tag{163}$$

where the reduced endpoint functions are

$$F_{\mathcal{O}_j}(x, x') \;:=\; \int_{\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}} \mathcal{O}_j(A)\,\mathbf{Q}^{x,x'}_{t,\mathrm{Reg}}(\mathrm{d}A). \tag{164}$$

An analogous formula holds for $\Phi_{\mathrm{full}}(G)$.

Step 3: conditional factorisation given B_0. Under the two-sided stationary Markov construction, conditional on B_0 the σ-algebras $\mathscr{F}_+$ and $\mathscr{F}_-$ are independent. Since F is $\mathscr{F}_+$-measurable and ΘG is $\mathscr{F}_-$-measurable, we have

$$\mathbb{E}_\infty\big[\overline{F}\,(\Theta G)\,\big|\,B_0\big] = \mathbb{E}_\infty\big[\overline{F}\,\big|\,B_0\big]\,\mathbb{E}_\infty\big[\Theta G\,\big|\,B_0\big] = \overline{\Phi_{\mathrm{full}}(F)(B_0)}\,\mathbb{E}_\infty\big[\Theta G\,\big|\,B_0\big].$$

Step 4: identify the reflected conditional expectation. By Lemma 13.5 and the fact that Θ fixes B_0 (see (157)), conditional expectations satisfy

$$\mathbb{E}_\infty\big[\Theta G\,\big|\,B_0\big] = \mathbb{E}_\infty\big[G \circ \Theta\,\big|\,B_0\big] = \mathbb{E}_\infty\big[G\,\big|\,B_0\big] = \Phi_{\mathrm{full}}(G)(B_0),$$

where the middle equality is the invariance statement at the level of regular conditional distributions given B_0.

Taking expectations in Step 3 now yields

$$\mathbb{E}_\infty\big[\overline{F}\,(\Theta G)\big] = \int_{\mathsf{H}} \overline{\Phi_{\mathrm{full}}(F)(x)}\;\Phi_{\mathrm{full}}(G)(x)\,\pi(\mathrm{d}x) = \langle \Phi_{\mathrm{full}}(F),\, \Phi_{\mathrm{full}}(G)\rangle_{L^2(\pi)}.$$

This proves (160) on the generating algebra, hence on all of $\mathcal{A}^{\mathrm{full}}_+$ by Step 1. Finally, (161) is the special case $G = F$. □

13.3 The full OS Hilbert space and time translation

Definition 13.7 (Full OS inner product and Hilbert space)**.** For $F, G \in \mathcal{A}^{\mathrm{full}}_+$ define

$$(F, G)_{\mathrm{OS}} \;:=\; \mathbb{E}_\infty\big[\overline{F}\,(\Theta G)\big]. \tag{165}$$

Let $\mathcal{N} := \{F \in \mathcal{A}^{\mathrm{full}}_+ \,:\, (F,F)_{\mathrm{OS}} = 0\}$. Define the pre-Hilbert space $\mathcal{D}_{\mathrm{OS}} := \mathcal{A}^{\mathrm{full}}_+/\mathcal{N}$ and its completion $\mathcal{H}^{\mathrm{full}}_{\mathrm{OS}}$.

Remark 13.8 (Identification with a subspace of $L^2(\pi)$). By Theorem 13.6, the map Φ_{full} descends to an isometric embedding of $\mathcal{H}_{\mathrm{OS}}^{\mathrm{full}}$ into $L^2(\pi)$:

$$(F,G)_{\mathrm{OS}} = \langle \Phi_{\mathrm{full}}(F), \Phi_{\mathrm{full}}(G)\rangle_{L^2(\pi)}.$$

Thus $\mathcal{H}_{\mathrm{OS}}^{\mathrm{full}}$ may be identified with the $L^2(\pi)$-closure of $\Phi_{\mathrm{full}}(\mathcal{A}_+^{\mathrm{full}})$.

Let S be the slab shift on Ω_{stack}:

$$(\mathsf{S}B)_n = B_{n+1}, \qquad (\mathsf{S}A)^{(n)} = A^{(n+1)}.$$

Define $\mathcal{T} : \mathcal{A}_+^{\mathrm{full}} \to \mathcal{A}_+^{\mathrm{full}}$ by $(\mathcal{T}F)(\omega) := F(\mathsf{S}\omega)$. This descends to an operator on $\mathcal{H}_{\mathrm{OS}}^{\mathrm{full}}$, still denoted $\mathcal{T}$.

Proposition 13.9 (Time translation corresponds to the transfer operator). *Under the identification of Remark 13.8, the OS time-translation operator $\mathcal{T}$ corresponds to the transfer operator T_t on $L^2(\pi)$:*

$$\Phi_{\mathrm{full}}(\mathcal{T}F) \;=\; \mathsf{T}_t\, \Phi_{\mathrm{full}}(F), \qquad F \in \mathcal{A}_+^{\mathrm{full}}. \tag{166}$$

In particular, $\mathcal{T}$ is a self-adjoint contraction on $\mathcal{H}_{\mathrm{OS}}^{\mathrm{full}}$.

Proof. Fix $F \in \mathcal{A}_+^{\mathrm{full}}$ and $x \in \mathsf{H}$. By definition and the Markov property of the boundary chain,

$$\begin{aligned}
\Phi_{\mathrm{full}}(\mathcal{T}F)(x) &= \mathbb{E}_\infty\big[F\circ \mathsf{S} \;\big|\; B_0 = x\big] \\
&= \int_{\mathsf{H}} \mathbb{E}_\infty\big[F \;\big|\; B_1 = x'\big]\; \mathsf{K}_t(x,\, \mathrm{d}x') \\
&= \int_{\mathsf{H}} \Phi_{\mathrm{full}}(F)(x')\, \mathsf{K}_t(x,\, \mathrm{d}x') = (\mathsf{T}_t \Phi_{\mathrm{full}}(F))(x).
\end{aligned}$$

which is (166). Self-adjointness and contractivity follow because T_t is a self-adjoint contraction on $L^2(\pi)$ (Definition 11.8 and Proposition 11.7), and Φ_{full} is an isometry. ☐

13.4 Correlation functions as transfer-operator matrix elements

Let $\mathbf{1}$ denote the constant function on H and also its class in $\mathcal{H}_{\mathrm{OS}}^{\mathrm{full}}$. This is the *vacuum vector*; indeed $\mathsf{T}_t\mathbf{1} = \mathbf{1}$ and $\mathcal{T}\mathbf{1} = \mathbf{1}$.

Define the mean-zero subspace

$$L_0^2(\pi) \;:=\; \Big\{ f \in L^2(\pi) :\; \langle f, \mathbf{1}\rangle_{L^2(\pi)} = 0 \Big\}, \tag{167}$$

and similarly $\mathcal{H}_{\mathrm{OS},0}^{\mathrm{full}} := \mathbf{1}^\perp \subset \mathcal{H}_{\mathrm{OS}}^{\mathrm{full}}$.

Lemma 13.10 (Stationary correlations and powers of T_t). *Let $(B_n)_{n\in\mathbb{Z}}$ be the stationary boundary chain with kernel K_t and invariant law π (Definition 11.12). Then for all $f, g \in L^2(\pi)$ and all $n \in \mathbb{N}$,*

$$\mathbb{E}_{\mathbb{P}_\pi^{\mathrm{tr}}}\big[f(B_0)\, g(B_n)\big] \;=\; \langle f, \mathsf{T}_t^n g\rangle_{L^2(\pi)}. \tag{168}$$

In particular, if $\pi(f) = 0 = \pi(g)$, then

$$\mathrm{Cov}_{\mathbb{P}_\pi^{\mathrm{tr}}}\big(f(B_0), g(B_n)\big) \;=\; \langle f, \mathsf{T}_t^n g\rangle_{L^2(\pi)}. \tag{169}$$

Proof. By the Markov property and stationarity,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_\pi^{\mathrm{tr}}}[f(B_0) g(B_n)] &= \mathbb{E}_{\mathbb{P}_\pi^{\mathrm{tr}}}\big[f(B_0)\, \mathbb{E}[g(B_n) \,|\, B_0]\big] \\
&= \mathbb{E}_{\mathbb{P}_\pi^{\mathrm{tr}}}\big[f(B_0)\, (\mathsf{T}_t^n g)(B_0)\big] \\
&= \int f(x)\, (\mathsf{T}_t^n g)(x)\, \pi(\,\mathrm{d}x) = \langle f, \mathsf{T}_t^n g\rangle_{L^2(\pi)}.
\end{aligned}$$

which is (168). If $\pi(f) = \pi(g) = 0$, then $\mathrm{Cov}_{\mathbb{P}_\pi^{\mathrm{tr}}}(f(B_0), g(B_n)) = \mathbb{E}_{\mathbb{P}_\pi^{\mathrm{tr}}}[f(B_0)g(B_n)]$, giving (169). ☐

13.5 From exponential clustering to a spectral gap

We now prove the key spectral statement. The logical structure is:

(i) Exponential clustering for a dense class of mean-zero observables forces the spectral measure of each such observable to be supported in $[-\rho, \rho]$.

(ii) Density then forces the entire spectrum of T_t on $L^2_0(\pi)$ to lie in $[-\rho, \rho]$.

(iii) The corresponding Euclidean mass parameter is $m_* = |\log \rho|/t$.

Assumption 13.11 (Density of bounded d-Lipschitz functions)**.** The set $\mathcal{L}_d \cap L^2(\pi)$ of bounded d-Lipschitz functions on H (Definition 9.9) is dense in $L^2(\pi)$.

Remark 13.12. A sufficient condition for Assumption 13.11 is that π has a strictly positive continuous density with respect to Lebesgue measure on H (which is finite-dimensional), in which case bounded C^1 functions with bounded gradient are dense in $L^2(\pi)$ and are in particular d-Lipschitz.

Lemma 13.13 (A convenient sufficient condition for Assumption 13.11)**.** *At fixed regulator, suppose H is finite-dimensional and $\pi(\,\mathrm{d}x) = p(x)\,\mathrm{d}x$ has a strictly positive C^1 density with respect to Lebesgue measure in a coordinate realisation of H. Then $\mathcal{L}_d \cap L^2(\pi)$ is dense in $L^2(\pi)$.*

Likewise, if $\mathsf{H} = G^N$ for a compact Lie group G equipped with a bi-invariant Riemannian metric and $\pi(\,\mathrm{d}x) = p(x)\,\mathrm{Haar}(\,\mathrm{d}x)$ has a strictly positive C^1 density with respect to Haar measure, then $\mathcal{L}_d \cap L^2(\pi)$ is dense in $L^2(\pi)$.

Corollary 13.14 (Density hypothesis in the Wilson corridor)**.** *In the Wilson finite-range lattice regulator family of Theorem 2.14, at fixed regulator the boundary law $\pi_{t,\mathrm{Reg}}$ has a strictly positive C^∞ density with respect to the Haar measure on G^{E_∂}. Hence Assumption 13.11 holds at fixed regulator by Lemma 13.13.*

Proof. At fixed regulator, the boundary state space is the compact manifold $\mathsf{B}_\partial = G^{E_\partial}$ (boundary links) with Haar reference measure. By definition, the boundary law is the endpoint marginal obtained by integrating the strictly positive Gibbs weight over the interior links:

$$\pi_{t,\mathrm{Reg}}(\,\mathrm{d}x) \;=\; \frac{1}{Z_{t,\mathrm{Reg}}}\Big(\int \exp\{-\beta S_{t,\mathrm{Reg}}(U)\}\,\mathrm{Haar}(\,\mathrm{d}U_{\mathrm{int}})\Big)\,\mathrm{Haar}(\,\mathrm{d}x).$$

Since $S_{t,\mathrm{Reg}}$ is a finite sum of smooth local terms on G^E, the map $x \mapsto \int \exp\{-\beta S_{t,\mathrm{Reg}}(U)\}\,\mathrm{Haar}(\,\mathrm{d}U_{\mathrm{int}})$ is C^∞ by dominated differentiation on the compact manifold, and it is strictly positive because the integrand is strictly positive everywhere. Hence $\pi_{t,\mathrm{Reg}}(\,\mathrm{d}x) = p_{t,\mathrm{Reg}}(x)\,\mathrm{Haar}(\,\mathrm{d}x)$ with $p_{t,\mathrm{Reg}} \in C^\infty(\mathsf{B}_\partial)$ and $p_{t,\mathrm{Reg}} > 0$. Lemma 13.13 then yields Assumption 13.11. $\square$

Proof of Lemma 13.13. In the Euclidean case, bounded C^1 functions with bounded gradient are d-Lipschitz and are dense in $L^2(\pi)$ by standard mollification and cutoff; see e.g. [2, 1]. For G^N, one may convolve with the heat kernel on G^N to approximate in $L^2(\pi)$ by smooth functions, and bounded smooth functions with bounded gradient are d-Lipschitz with respect to the product geodesic metric; see e.g. [10]. $\square$

Lemma 13.15 (Finite-dimensional verification of Assumption 13.11)**.** *Suppose that, at fixed regulator, the boundary space H is finite-dimensional and π admits a strictly positive C^1 density with respect to Lebesgue measure (or, on a compact manifold such as G^N, with respect to the corresponding Haar volume form). Then Assumption 13.11 holds.*

Proof. In finite dimension, smooth compactly supported functions are dense in $L^2(\pi)$ under the stated non-degeneracy of the density. Indeed, let $f \in L^2(\pi)$; truncate f to a bounded function and localise to a large ball (or use a partition of unity on a compact manifold) to reduce to $f \in L^2(p\, dx)$ with $p \in C^1$ and $p > 0$ on the localisation domain. Standard mollification then yields $f_k \in C^\infty$ with $f_k \to f$ in $L^2(\pi)$. Finally, truncate f_k to be bounded; bounded C^1 functions are d-Lipschitz for any bounded cost d that dominates the Euclidean distance on bounded sets, hence f is approximated in $L^2(\pi)$ by bounded d-Lipschitz functions. $\square$

Remark 13.16 (Comments on Assumption 13.11). At fixed finite-dimensional H, Assumption 13.11 is automatic under mild non-degeneracy: if π has a strictly positive density with respect to a smooth reference (Lebesgue on $\mathbb{R}^d$ or Haar on a compact manifold), then smooth bounded functions are dense in $L^2(\pi)$ and can be approximated by bounded Lipschitz functions by mollification in local charts. The assumption can fail only if π is supported on a lower-dimensional subset or has genuine singular components; in gauge contexts this is precisely why one either gauge-fixes (so H is a genuine coordinate space) or formulates the transfer theory on a smooth gauge-invariant state space with a non-degenerate reference measure.

Lemma 13.17 (Autocorrelation decay for centered bounded d-Lipschitz functions)**.** *Assume the boundary mixing estimate* (85) *holds with constants* (C, ρ) *and assume* $\pi(V) < \infty$*. Let* $f \in \mathcal{L}_d$ *and assume* $\pi(f) = 0$*. Then for all* $n \in \mathbb{N}$*,*

$$\left|\langle f, \mathsf{T}_t^n f\rangle_{L^2(\pi)}\right| = \left|\mathrm{Cov}_{\mathbb{P}_\pi^{\mathrm{tr}}}(f(B_0), f(B_n))\right| \;\leq\; \|f\|_{L^\infty(\mathsf{H})} \,\mathrm{Lip}_d(f)\, C\, \rho^n\, \pi(V). \tag{170}$$

In particular, for even powers,

$$0 \;\leq\; \langle f, \mathsf{T}_t^{2n} f\rangle_{L^2(\pi)} \;\leq\; \|f\|_{L^\infty} \,\mathrm{Lip}_d(f)\, C\, \rho^{2n}\, \pi(V), \qquad n \in \mathbb{N}. \tag{171}$$

Proof. The equality in (170) is Lemma 13.10 with $g = f$ and $\pi(f) = 0$. The bound follows from the boundary time-axis exponential clustering two-point estimate (e.g. (110)) applied with $F = f(B_0)$ and $G = f(B_n)$. For (171), note that T_t is self-adjoint, hence $\mathsf{T}_t^{2n} = (\mathsf{T}_t^n)^*\mathsf{T}_t^n$ is positive semidefinite, so $\langle f, \mathsf{T}_t^{2n} f\rangle \geq 0$ for all $f \in L^2(\pi)$. The upper bound is (170) with $n \mapsto 2n$. $\square$

Let $\mathsf{E}(\cdot)$ denote the spectral resolution of the self-adjoint contraction T_t on $L^2(\pi)$. For $f \in L^2(\pi)$, define the associated spectral measure

$$\mu_f(A) \;:=\; \langle f, \mathsf{E}(A) f\rangle_{L^2(\pi)}, \qquad A \subset [-1,1] \text{ Borel}. \tag{172}$$

Then μ_f is a finite positive Borel measure and, for $n \in \mathbb{N}$,

$$\langle f, \mathsf{T}_t^n f\rangle_{L^2(\pi)} \;=\; \int_{[-1,1]} \lambda^n\, \mu_f(\mathrm{d}\lambda). \tag{173}$$

Lemma 13.18 (Support bound from exponential autocorrelation decay)**.** *Assume there exists* $\rho \in (0,1)$ *such that for a given nonzero* $f \in L^2_0(\pi)$ *there exists a constant* $C_f < \infty$ *with*

$$\langle f, \mathsf{T}_t^{2n} f\rangle_{L^2(\pi)} \;\leq\; C_f\, \rho^{2n} \qquad \textit{for all } n \in \mathbb{N}. \tag{174}$$

Then the spectral measure μ_f *is supported in* $[-\rho, \rho]$*, i.e.*

$$\mu_f(\{\lambda : \; |\lambda| > \rho\}) = 0. \tag{175}$$

Proof. Assume for contradiction that $\mu_f(\{|\lambda| > \rho\}) > 0$. Then there exists $\varepsilon > 0$ such that

$$\mu_f(\{|\lambda| \geq \rho + \varepsilon\}) \ > \ 0,$$

because $\{|\lambda| > \rho\} = \bigcup_{k\in\mathbb{N}}\{|\lambda| \geq \rho + 1/k\}$ and μ_f is finite.

Using (173) with even powers,

$$\langle f, \mathsf{T}_t^{2n} f\rangle = \int \lambda^{2n}\,\mu_f(\mathrm{d}\lambda) \ \geq \ \int_{\{|\lambda|\geq\rho+\varepsilon\}} \lambda^{2n}\,\mu_f(\mathrm{d}\lambda) \ \geq \ (\rho+\varepsilon)^{2n}\,\mu_f(\{|\lambda| \geq \rho+\varepsilon\}).$$

Dividing by ρ^{2n} gives

$$\frac{\langle f, \mathsf{T}_t^{2n} f\rangle}{\rho^{2n}} \ \geq \ \Big(1+\frac{\varepsilon}{\rho}\Big)^{2n}\,\mu_f(\{|\lambda| \geq \rho+\varepsilon\}) \ \xrightarrow[n\to\infty]{} \ \infty,$$

which contradicts (174). Therefore (175) holds. □

Theorem 13.19 (Spectral gap for T_t from Harris/time-axis exponential clustering)**.** *Assume:*

(i) *the boundary mixing estimate* (85) *holds with constants* (C,ρ) *for some* $\rho \in (0,1)$*;*

(ii) $\pi(V) < \infty$*;*

(iii) *Assumption 13.11 holds (bounded d-Lipschitz functions are dense in* $L^2(\pi)$*).*

Then the spectrum of T_t *on* $L^2_0(\pi)$ *is contained in* $[-\rho,\rho]$*:*

$$\sigma(\mathsf{T}_t|_{L^2_0(\pi)}) \ \subset \ [-\rho,\rho]. \tag{176}$$

Equivalently, T_t *has a spectral gap above the vacuum eigenvalue* 1*, and the corresponding Euclidean time mass parameter satisfies*

$$m_* \ := \ \frac{|\log\rho|}{t} \ \leq \ -\frac{1}{t}\log\Big(\sup\{|\lambda| : \ \lambda\in\sigma(\mathsf{T}_t)\setminus\{1\}\}\Big). \tag{177}$$

Under the identification of Proposition 13.9, the same spectral inclusion holds for the OS time-translation operator $\mathcal{T}$ *on* $\mathcal{H}^{\mathrm{full}}_{\mathrm{OS},0}$*.*

Proof. Step 1: support bound for a dense set. Let $f \in \mathcal{L}_d \cap L^2_0(\pi)$. By Lemma 13.17, f satisfies (174) with

$$C_f := \|f\|_\infty\,\mathrm{Lip}_d(f)\,C\,\pi(V),$$

hence by Lemma 13.18 its spectral measure μ_f is supported in $[-\rho,\rho]$.

Step 2: eliminate spectral mass outside $[-\rho,\rho]$. Let $E := \{\lambda : \ |\lambda| > \rho\} \subset [-1,1]$ and let $\mathsf{E}(E)$ be the corresponding spectral projection. Suppose for contradiction that $\mathsf{E}(E) \neq 0$ on $L^2_0(\pi)$. Then the range $\mathsf{E}(E)L^2_0(\pi)$ is a nonzero closed subspace of $L^2_0(\pi)$. By Assumption 13.11, $\mathcal{L}_d \cap L^2_0(\pi)$ is dense in $L^2_0(\pi)$, hence intersects this subspace: there exists $f \in \mathcal{L}_d \cap L^2_0(\pi)$ with $\mathsf{E}(E)f \neq 0$.

But for such f,

$$\mu_f(E) = \langle f, \mathsf{E}(E)f\rangle = \|\mathsf{E}(E)f\|^2_{L^2(\pi)} \ > \ 0,$$

which contradicts Step 1. Therefore $\mathsf{E}(E) = 0$ on $L^2_0(\pi)$, which is exactly (176).

Step 3: gap and mass parameter. Since T_t is a self-adjoint contraction with eigenvalue 1 (constants), (176) implies

$$\sup\{|\lambda| : \lambda\in\sigma(\mathsf{T}_t)\setminus\{1\}\} \ \leq \ \rho.$$

Taking $-\frac{1}{t}\log(\cdot)$ gives (177). Finally, Proposition 13.9 identifies $\mathcal{T}$ with T_t on the embedded OS space, so the same spectral inclusion holds on $\mathcal{H}^{\mathrm{full}}_{\mathrm{OS},0}$. □

Remark 13.20 (What was proved, precisely). At fixed regulator, the Harris/time-axis exponential clustering mixing rate ρ bounds the *entire nontrivial spectrum* of the Euclidean time transfer operator T_t inside $[-\rho, \rho]$, provided bounded d-Lipschitz functions are dense in $L^2(\pi)$ (Assumption 13.11). This yields a checkable "gap" statement in the transfer-operator sense, with Euclidean mass parameter $m_* = |\log \rho|/t$.

14 Regulator-uniform consequences and main theorem

This section isolates which inputs must be uniform in the regulator Reg and then states the main regulator-uniform conclusions that follow *within the scope of this paper*. We record uniform exponential clustering in Euclidean time for a concrete class of bounded gauge-invariant slab observables and the corresponding uniform spectral gap bound for the Euclidean-time transfer operator.

In this paper we *do* construct two genuine limiting objects within the strong-coupling slab framework: (a) a UV-refinement (continuum-in-the-box) projective limit at fixed (t, L) for the boundary/end-point laws and the associated finite-window transfer specifications, and (b) a spatial thermodynamic limit $L \to \infty$ in an L-uniform KP corridor for local observables and the induced boundary/transfer objects (Section I). We do *not* claim a full renormalised continuum Yang–Mills theory on $\mathbb{R}^4$ (or a Clay-style mass gap) beyond the time-axis OS/transfer framework, and we do not address limits outside the strong-coupling corridor.

14.1 Uniformity requirements: the constants ledger

Throughout, $t > 0$, $L > 0$, and $\delta \in (0, t/2)$ are fixed, and Reg ranges over a family of regulators (e.g. Fourier cutoff $\Lambda \to \infty$ together with the fixed gauge-fixing regularisation scheme of Sections 4–11).

For each Reg we have:

- a boundary trace space $\mathsf{H} = \mathsf{H}_{\mathrm{Reg}}$ (finite-dimensional) and a stationary boundary chain with kernel $\mathsf{K}_t^{(\mathrm{Reg})}$ and invariant law $\pi^{(\mathrm{Reg})}$ (Section 11);
- a Lyapunov function $V^{(\mathrm{Reg})} : \mathsf{H} \to [1, \infty)$ and geometric ergodicity constants for the boundary chain (Sections 7–8);
- a family of one-slab conditional interior measures $\mathbf{Q}_{t,\mathrm{Reg}}^{b^-,b^+}$ and response objects $\mathscr{J}_{t,\mathrm{Reg}}(b^-, b^+; A)$ controlling endpoint derivatives of reduced observables (Sections 4–5);
- the stacked/DLR Euclidean measure $\mathbf{P}_{t,\mathrm{Reg}}^{\infty}$ and transfer operator $\mathsf{T}_t^{(\mathrm{Reg})}$ on $L^2(\pi^{(\mathrm{Reg})})$ (Sections 11 and 13).

The regulator-uniform conclusions require uniform bounds on the following quantities.

Assumption 14.1 (Uniform constants ledger)**.** There exist constants

$$C_* \in (0, \infty), \qquad \rho_* \in (0, 1), \qquad M_{V,*} \in (0, \infty), \qquad M_{\mathscr{J},*}(t, \delta) \in (0, \infty),$$

such that for every regulator Reg:

(i) **Uniform geometric ergodicity:** the boundary chain satisfies the V-weighted geometric ergodicity estimate (85) with

$$C^{(\mathrm{Reg})} \;\le\; C_*, \qquad \rho^{(\mathrm{Reg})} \;\le\; \rho_*. \tag{178}$$

(ii) **Uniform Lyapunov moment:** the invariant law satisfies

$$\pi^{(\mathrm{Reg})}\big(V^{(\mathrm{Reg})}\big) \;\le\; M_{V,*}. \tag{179}$$

(iii) **Uniform response moment (interior support):** for the response density $\mathscr{J}_{t,\mathrm{Reg}}$ used in Section 12,

$$\sup_{\mathrm{Reg}} \;\sup_{b^-,b^+\in \mathsf{H}_{\mathrm{Reg}}} \;\mathbb{E}_{\mathbf{Q}^{b^-,b^+}_{t,\mathrm{Reg}}}\big[\|\mathscr{J}_{t,\mathrm{Reg}}(b^-,b^+;A)\|_{\mathsf{H}_{\mathrm{Reg}}}\big] \;\le\; M_{\mathscr{J},*}(t,\delta). \tag{180}$$

Corollary 14.2 (Ledger constants in the Wilson lattice corridor)**.** *In the Wilson lattice regulator family of Theorem 2.14 (fixed (t,L), UV refinement and auxiliary truncations), the transfer-side inputs in Assumption 14.1 that concern (a) cylindrical transfer-kernel contraction and (b) transfer-side mLSI/L^2 mixing hold with constants provided by Corollary 2.29 (Appendix F) and Corollary 2.25 (Appendices G–H), respectively. Moreover, at each fixed regulator the density condition Assumption 13.11 holds under the mild sufficient hypothesis of Lemma 13.13 (e.g. a strictly positive C^1 density with respect to Lebesgue/Haar in a finite-dimensional realisation). If, in addition, the uniform response moment bound Assumption 12.6 holds in this lattice corridor for the class of bulk observables under consideration, then the remaining response constants in Assumption 14.1 are available and the full conclusion package of Theorem 14.8 applies.*

Remark 14.3 (What is *not* assumed here). Assumption 14.1 does *not* assume any regulator-uniform control of:

- spatial infinite-volume limits $L\to\infty$ (except for the strong-coupling/KP corridor result of Section I);
- renormalised continuum limits beyond the chosen regulator scheme;
- OS reconstruction in the full $\mathbb{R}^4$ sense (only the Euclidean time-axis transfer-operator OS structure is used here);
- gauge-fixing independence.

It is a statement about uniformity *within* the regulated slab-and-transfer setup of this paper.

14.2 Regulator-uniform exponential clustering for gauge-invariant observables

Recall the gauge-invariant admissible class $\mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$ (Definition 12.3) and the connected two-point Schwinger function $S^{\mathrm{conn}}_{\mathcal{O},\mathcal{P}}(n)$ (Definition 12.10).

Theorem 14.4 (Uniform two-point clustering for gauge-invariant slab observables)**.** *Fix $t>0$, $L>0$ and $\delta\in(0,t/2)$. Assume Assumption 14.1. Then for every regulator* Reg*, every $\mathcal{O},\mathcal{P}\in\mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$ (defined on $\mathsf{X}^{\mathrm{temp}}_{t,\mathrm{Reg}}$), and every integer $n\ge 2$,*

$$\big|S^{\mathrm{conn},(\mathrm{Reg})}_{\mathcal{O},\mathcal{P}}(n)\big| \;\le\; \|\mathcal{O}\|_\infty\ \max\Big\{\frac{2\|\mathcal{P}\|_\infty\, M_{\mathscr{J},*}(t,\delta)}{\alpha},\ 2\|\mathcal{P}\|_\infty\Big\}\, C_*\,\rho_*^{n-1}\, M_{V,*}. \tag{181}$$

Equivalently, writing $s=nt$ and $m_*:=|\log\rho_*|/t$,

$$\big|S^{\mathrm{conn},(\mathrm{Reg})}_{\mathcal{O},\mathcal{P}}(n)\big| \;\le\; \mathsf{C}_{\mathcal{O},\mathcal{P},*}(t,\delta)\,\exp(-m_* s), \tag{182}$$

where

$$\mathsf{C}_{\mathcal{O},\mathcal{P},*}(t,\delta) \;:=\; \|\mathcal{O}\|_\infty\ \max\Big\{\frac{2\|\mathcal{P}\|_\infty\, M_{\mathscr{J},*}(t,\delta)}{\alpha},\ 2\|\mathcal{P}\|_\infty\Big\}\, C_*\,\rho_*^{-1}\, M_{V,*}. \tag{183}$$

In particular, the decay rate m_* and the constants in (182) are independent of Reg.

Proof. Theorem 12.11 holds for each fixed regulator with constants $(C^{(\mathrm{Reg})}, \rho^{(\mathrm{Reg})})$, $\pi^{(\mathrm{Reg})}(V^{(\mathrm{Reg})})$, and $M^{(\mathrm{Reg})}_{\mathscr{J}}(t,\delta)$. Under Assumption 14.1, these quantities are uniformly bounded by (C_*, ρ_*), $M_{V,*}$, and $M_{\mathscr{J},*}(t,\delta)$ respectively. Substituting these bounds into (151) yields (181). The exponential-in-s form (182) follows exactly as in the proof of Theorem 12.11. $\square$

Corollary 14.5 (Uniform clustering for Wilson loops)**.** *Under the assumptions of Theorem 14.4, for any loops $\gamma_0, \gamma_1 \subset S^{(\delta)}_{t,L}$ and unitary representations ϱ_0, ϱ_1 of G, the Wilson-loop connected Schwinger function satisfies, for all $n \geq 2$,*

$$\left|\mathrm{Cov}^{(\mathrm{Reg})}_{\infty}(\mathcal{W}^{[0]}_{\gamma_0,\varrho_0}, \mathcal{W}^{[n]}_{\gamma_1,\varrho_1})\right| \;\leq\; \max\Big\{\frac{2\,M_{\mathscr{J},*}(t,\delta)}{\alpha},\, 2\Big\}\, C_*\, \rho_*^{n-1}\, M_{V,*},$$

using $\|\mathcal{W}_{\gamma,\varrho}\|_\infty \leq 1$.

14.3 Regulator-uniform transfer-operator gap

We now record the corresponding uniform spectral inclusion for the Euclidean-time transfer operator. For each regulator, $\mathsf{T}^{(\mathrm{Reg})}_t$ is a self-adjoint contraction on $L^2(\pi^{(\mathrm{Reg})})$ (Section 11) with eigenvalue 1 corresponding to constants.

To formulate a uniform gap statement, we require the density hypothesis (Assumption 13.11) regulator-by-regulator. Since $\mathsf{H}_{\mathrm{Reg}}$ varies with Reg, we record it as a uniform assumption.

Assumption 14.6 (Uniform density of bounded d-Lipschitz functions)**.** For each regulator Reg, bounded d-Lipschitz functions on $\mathsf{H}_{\mathrm{Reg}}$ are dense in $L^2(\pi^{(\mathrm{Reg})})$.

Theorem 14.7 (Uniform spectral gap bound for $\mathsf{T}^{(\mathrm{Reg})}_t$)**.** *Fix $t > 0$ and assume Assumptions 14.1 and 14.6. Then for every regulator* Reg*,*

$$\sigma\big(\mathsf{T}^{(\mathrm{Reg})}_t|_{L^2_0(\pi^{(\mathrm{Reg})})}\big) \;\subset\; [-\rho_*, \rho_*]. \tag{184}$$

Equivalently,

$$\sup\{|\lambda| :\; \lambda \in \sigma(\mathsf{T}^{(\mathrm{Reg})}_t) \setminus \{1\}\} \;\leq\; \rho_*, \tag{185}$$

uniformly in Reg*, and the associated Euclidean mass parameter satisfies the uniform lower bound*

$$m_* \;:=\; \frac{|\log \rho_*|}{t} \;\leq\; -\frac{1}{t}\log\Big(\sup\{|\lambda| :\; \lambda \in \sigma(\mathsf{T}^{(\mathrm{Reg})}_t) \setminus \{1\}\}\Big). \tag{186}$$

Proof. Apply Theorem 13.19 for each fixed regulator, using the mixing constants $C^{(\mathrm{Reg})}, \rho^{(\mathrm{Reg})}$ and the Lyapunov moment bound $\pi^{(\mathrm{Reg})}(V^{(\mathrm{Reg})})$. The proof of Theorem 13.19 uses only: (i) the geometric covariance decay estimate for bounded d-Lipschitz functions with constants $(C^{(\mathrm{Reg})}, \rho^{(\mathrm{Reg})})$, (ii) finiteness of $\pi^{(\mathrm{Reg})}(V^{(\mathrm{Reg})})$, and (iii) density of bounded d-Lipschitz functions in $L^2(\pi^{(\mathrm{Reg})})$. Assumptions 14.1 and 14.6 ensure these hypotheses hold uniformly and with $\rho^{(\mathrm{Reg})} \leq \rho_*$. Hence $\sigma(\mathsf{T}^{(\mathrm{Reg})}_t|_{L^2_0}) \subset [-\rho^{(\mathrm{Reg})}, \rho^{(\mathrm{Reg})}] \subset [-\rho_*, \rho_*]$, which is (184). The equivalent formulations (185) and (186) follow exactly as in Theorem 13.19. $\square$

14.4 Main theorem of this paper: what survives uniformly and what is not claimed

We now package the regulator-uniform conclusions in a single statement. To keep scope disciplined, the theorem is stated entirely within the regulated slab/transfer-operator framework developed above.

Theorem 14.8 (Regulator-uniform Euclidean-time exponential clustering and transfer gap)**.** *Fix $t > 0$, $L > 0$ and $\delta \in (0, t/2)$. Assume Assumption 14.1. Then for each regulator* Reg*, the bi-infinite Euclidean-time DLR measure $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$ (Section 11) and the associated self-adjoint transfer operator $\mathsf{T}_t^{(\mathrm{Reg})}$ on $L^2(\pi^{(\mathrm{Reg})})$ are well-defined, and the following hold:*

(i) ***(Uniform time clustering for gauge-invariant observables)*** *For all $\mathcal{O}, \mathcal{P} \in \mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$ and all $n \geq 2$, the connected two-point Schwinger function satisfies (181)–(182), with decay rate $m_* = |\log \rho_*|/t$ and prefactor $\mathsf{C}_{\mathcal{O},\mathcal{P},*}(t, \delta)$ independent of* Reg*.*

(ii) ***(OS time-axis reconstruction)*** *The time reflection Θ induces a reflection-positive OS Hilbert space $\mathcal{H}^{\mathrm{full}}_{\mathrm{OS}}$ (Section 13), which embeds isometrically into $L^2(\pi^{(\mathrm{Reg})})$ via Φ_{full}, and the Euclidean time translation by one slab corresponds to $\mathsf{T}_t^{(\mathrm{Reg})}$.*

(iii) ***(Uniform transfer-operator gap, conditional on density)*** *If, in addition, Assumption 14.6 holds, then the nontrivial spectrum of $\mathsf{T}_t^{(\mathrm{Reg})}$ is uniformly bounded by ρ_* as in (184), hence the transfer-operator mass parameter satisfies (186).*

Remark 14.9 (Interpretation and limitations)*.* Theorem 14.8 is an *Euclidean-time* statement for a regulated slab construction: it provides regulator-uniform exponential clustering for a concrete class of bounded gauge-invariant interior-supported cylinder slab observables (namely $\mathfrak{Obs}^{\mathrm{temp}}_{t,\delta}$) supported away from time boundaries, and the corresponding uniform spectral gap bound for the Euclidean-time transfer operator (under a density hypothesis).

The theorem does *not* claim:

- convergence of $\mathbf{P}^{\infty}_{t,\mathrm{Reg}}$ as $\mathrm{Reg} \to \infty$ *without* an additional projective/tightness input (see Subsection 14.5);
- existence of a continuum, renormalised $\mathbb{R}^4$ Euclidean Yang–Mills measure;
- spatial infinite-volume limits outside the KP corridor (Section I constructs $L \to \infty$ within an L-uniform corridor for local observables), or Wilson-loop area laws;
- uniqueness of a reconstructed Minkowski theory beyond the time-axis transfer framework.

These are separate analytical problems. The purpose of the present paper is to make the *Harris-mixing-to-time-axis exponential clustering engine* auditable and to isolate the precise uniform estimates required for any further limiting procedure.

14.5 A UV-refinement limit step at fixed (t, L)

The results of this paper are formulated at a fixed regulator Reg (fixed ultraviolet cutoff/refinement and any auxiliary truncations), with quantitative bounds that are *uniform* under UV refinement at fixed (t, L). One may ask for at least one genuine limiting construction *within the scope used here*. We therefore record a minimal UV-limit step that is sufficient for the *window-level* (cylindrical) transfer/OS mechanism driven by Appendix F.

Setup. Fix (t, L) and a cofinal UV-refinement sequence $(\mathrm{Reg}_n)_{n\geq 0}$ (for example, lattice spacing $a_n \downarrow 0$ or cutoff $\Lambda_n \uparrow \infty$) within the same finite slab geometry. For each n, let π_n be the one-boundary law and ν_n the two-boundary endpoint law, so that $\nu_n(\mathrm{d}b_0, \mathrm{d}b_1) = \pi_n(\mathrm{d}b_0)\, K_{t,n}(b_0, \mathrm{d}b_1)$ defines the transfer kernel $K_{t,n}$ by disintegration.

Fix a *finite window* W of boundary degrees of freedom that is stable along the refinement sequence, in the following sense: there exists a compact metric space X_W and, for all sufficiently large n, a continuous map $\Pi_W^{(n)} : \mathsf{X}_n \to \mathsf{X}_W$ (a coordinate restriction, block variable, or any fixed finite observable map) such that all W–cylindrical observables of interest factor through $\Pi_W^{(n)}$.

Define the projected laws on the fixed compact space X_W by

$$\pi_n^W \ := \ (\Pi_W^{(n)})_\# \pi_n, \qquad \nu_n^W \ := \ (\Pi_W^{(n)} \times \Pi_W^{(n)})_\# \nu_n.$$

Proposition 14.10 (Window-level UV-limit along a subsequence)**.** *For each fixed window W, the family $\{\pi_n^W : n \geq n_0(W)\}$ is tight on X_W and hence relatively compact for weak convergence. Likewise, $\{\nu_n^W : n \geq n_0(W)\}$ is tight on $\mathsf{X}_W \times \mathsf{X}_W$. Consequently, for every refinement sequence* (Reg_n) *there exists a subsequence (n_j) and probability measures π_∞^W on X_W and ν_∞^W on $\mathsf{X}_W \times \mathsf{X}_W$ such that*

$$\pi_{n_j}^W \ \Rightarrow \ \pi_\infty^W, \qquad \nu_{n_j}^W \ \Rightarrow \ \nu_\infty^W, \qquad j \to \infty.$$

Moreover, since X_W is a standard Borel space, ν_∞^W admits a regular conditional distribution, i.e. there exists a kernel $K_{t,\infty}^W$ on X_W such that $\nu_\infty^W(\mathrm{d}b_0, \mathrm{d}b_1) = \pi_\infty^W(\mathrm{d}b_0)\, K_{t,\infty}^W(b_0, \mathrm{d}b_1)$.

Proof. Because X_W is compact metric, every family of probability measures on X_W (and on $\mathsf{X}_W \times \mathsf{X}_W$) is tight. By Prokhorov's theorem this implies relative compactness for weak convergence, hence the existence of a convergent subsequence for π_n^W and for ν_n^W. Existence of a disintegration kernel $K_{t,\infty}^W$ for ν_∞^W with respect to its first marginal is standard on compact metric spaces. $\square$

Passing uniform window estimates to the limit. If Appendix F provides a projected Doeblin minorisation on W with constants *uniform in* n (at fixed (t, L)), then the same minorisation holds for the limiting endpoint law ν_∞^W and kernel $K_{t,\infty}^W$ by weak convergence and the Portmanteau theorem. In particular, any window-level geometric contraction and the resulting time-axis exponential clustering bounds for W–cylindrical observables persist along the UV-limit subsequence produced above. This is a genuine (though window-level and subsequential) UV-refinement limit construction within the scope of the present manuscript: it produces limiting objects for the class of observables for which the paper proves uniform transfer/OS estimates, without claiming a full continuum or $L \to \infty$ theory.

14.6 On charts and global derivatives in the Wilson corridor

In the Wilson lattice specialization with compact gauge group G, no auxiliary "small-field"/chart cutoff is required: the configuration space is compact and the local plaquette potentials are smooth globally. All derivative bounds in the KP/cluster-expansion corridor can be formulated using left/right-invariant directional derivatives on G, so the corridor results in Sections G–H and Appendix H are stated and proved without restricting plaquettes to a chart ball.

14.7 How the KP corridor threshold depends on t and L

The KP corridor requires $0 < \beta \leq \beta_\star(t, L)$. Most results in the paper are stated for fixed (t, L); however Section I proves a thermodynamic-limit result $L \to \infty$ within an L-uniform corridor $0 < \beta \leq \beta_\star^\infty(t)$. It is also useful to note how the corridor threshold behaves as t and L vary.

From the semiconvexity requirement in Theorem 2.23 and the KP Hessian bound (219), a necessary condition is

$$C_{\mathrm{Hess}}(t, L)\, \beta \ < \ \frac{2}{t},$$

so one may always take

$$\beta_\star(t,L) \;\le\; \frac{2}{t\,C_{\mathrm{Hess}}(t,L)}.$$

Thus, unless $C_{\mathrm{Hess}}(t,L)$ decays in t, the corridor typically shrinks at least like t^{-1} as $t \to \infty$ (equivalently, in terms of g, one requires $g \gtrsim t^{1/2}$). For $L \to \infty$, maintaining a nontrivial corridor would require that the KP majorant and polymer-counting constants entering Appendix H are *uniform in volume* (a genuine infinite-volume cluster-expansion input). Outside the L-uniform corridor, this paper does not address that regime; within an L-uniform strong-coupling corridor the thermodynamic limit $L \to \infty$ is constructed in Section I. The estimates in the fixed-(t,L) corridor are uniform only in UV refinement and auxiliary truncations.

14.8 Relationship to the standard OS axioms

OS axioms in brief. In the classical OS framework, one starts with a Euclidean field theory specified by a family of Schwinger functions (or, equivalently, a probability measure on a suitable space of distributions), and imposes a collection of axioms including Euclidean invariance, reflection positivity, symmetry, and regularity/growth conditions. These axioms ensure reconstruction of a Wightman theory with a Hilbert space, a vacuum vector, and a unitary representation of the Poincaré group, together with a positive energy condition. Exponential clustering of Schwinger functions in Euclidean time is then linked to a positive mass gap.

What is implemented here. The present paper constructs and uses a *time-axis* version of the OS machinery adapted to a slab decomposition. The reconstruction carried out in Section 13 uses:

- **Reflection positivity in Euclidean time:** a reflection map Θ about the time-0 slice (Definition 13.1) and positivity of the pairing $\mathbb{E}_\infty[\overline{F}\,(\Theta F)]$ for F in a positive-time cylinder algebra (Theorem 13.6). This is the OS reflection-positivity axiom specialised to the time direction and to a cylinder algebra naturally adapted to a transfer-operator setting.
- **Time translation invariance:** stationarity of the stacked/DLR measure under the slab shift S, which yields a well-defined time translation operator $\mathcal{T}$ on the reconstructed Hilbert space.
- **A concrete transfer operator:** identification of $\mathcal{T}$ with the Markov transfer operator T_t on $L^2(\pi)$ (Proposition 13.9), where π is the stationary boundary trace law induced by the slab weight.

Within this setting, "OS reconstruction" is not merely an abstract existence theorem: it is an explicit and checkable representation of Euclidean time translations by an operator acting on $L^2(\pi)$, via the map $F \mapsto \Phi_{\mathrm{full}}(F)$.

time-axis exponential clustering as a clustering property (time-axis version). The label time-axis exponential clustering is used here in the constructive-QFT sense of a quantitative clustering/ergodicity condition along Euclidean time translations (i.e. the OS clustering axiom restricted to the time direction, at regulator level). What is proved is a family of exponential covariance bounds for (a) boundary observables (Section 9), and (b) gauge-invariant slab observables supported away from time boundaries (Section 12), both with explicit dependence on the constants ledger. This is the kind of statement needed in constructive approaches where one aims to control long-time correlations uniformly across regulators and then interpret the decay rate as a mass parameter in the time-axis transfer sense.

Compatibility with the full OS axioms. Nothing in the paper contradicts the standard OS framework. Rather, the paper works in a deliberately *restricted direction*: it focuses on reflection positivity and translation/clustering in *one* Euclidean direction (time), leaving other Euclidean symmetries and continuum regularity issues outside the scope. From the OS perspective, the results verify a substantial fragment of the axiomatics at regulator level: reflection positivity in the time direction together with quantitative (time-axis) clustering for a robust class of observables.

14.9 Relationship to constructive QFT practice

Why a slab/transfer picture is natural. In many constructive and probabilistic approaches, one exploits a Euclidean-time decomposition to obtain an operator-theoretic description: finite-time Euclidean weights produce a transfer operator, and reflection positivity ensures that this operator is realised on a physical Hilbert space. In lattice or continuum regularised settings, this is the technical bridge between Euclidean probability and quantum Hamiltonians.

The present paper follows this paradigm, but with two emphases:

(i) **Quantitative ergodicity as a primary input.** Rather than attempting to prove clustering directly by multiscale expansion of Schwinger functions, the paper routes through a Harris-type drift/minorisation mechanism for a boundary Markov chain. This is designed to be robust under regulator changes and to yield explicit geometric rates.

(ii) **Auditability of endpoint dependence.** For slab observables, the main technical work is to control how interior expectations depend on boundary data. This is captured by response identities and moment bounds for the response density, allowing one to transport boundary mixing into slab mixing with explicit constants.

This structure matches constructive methodology: isolate a small set of verifiable, regulator-robust estimates and build long-distance physics (clustering, transfer-operator gaps) as a consequence.

Why bounded gauge-invariant observables. The clustering theorem in Section 12 is stated for bounded gauge-invariant observables supported away from time boundaries. This choice is deliberate. Boundedness ensures integrability without additional tail conditions. Interior support enables quasi-local control of boundary influence via response kernels. Gauge invariance ensures the observables are physically meaningful and avoids dependence on gauge-fixing details at the level of *observables*. Within this class, the paper proves Euclidean-time clustering with explicit constants and (under a density hypothesis) a corresponding transfer-operator gap.

Role of the "uniformity ledger." Section 14 makes explicit that only a finite set of quantities must be controlled uniformly in Reg: geometric ergodicity constants, a Lyapunov moment bound, and response moments for interior-supported observables. This is typical constructive-QFT hygiene: if a continuum limit is to be taken later, one must know exactly which estimates need to be stable.

14.10 What remains for a full continuum Euclidean theory

The results of this paper are formulated at fixed $t > 0$ and fixed spatial torus $\mathbb{T}^3_L$. They are therefore best understood as *uniform Euclidean-time estimates for a family of regulated theories*. To obtain a full continuum Euclidean QFT on $\mathbb{R}^4$ in the traditional sense, additional steps are required. We list them cautiously, since the appropriate strategy depends on the regulator scheme.

(A) A continuum limit as $\mathrm{Reg} \to \infty$**.** Theorems 14.4 and 14.7 provide *uniform bounds*, but they do not by themselves produce a limiting measure. A continuum Euclidean measure would require, for example:

- tightness (or another compactness mechanism) of the family of regulated measures on suitable distribution spaces;
- an identification of the limiting object under the chosen regulator (e.g. projective consistency, or another reconstruction mechanism);
- control of renormalisation and counterterms compatible with gauge symmetry (within the chosen scheme).

None of these are attempted here. The point of the present paper is that if such a limit exists along some subsequence, the uniform bounds would be available to pass to the limit and yield Euclidean-time clustering and a time-axis mass parameter in that limit.

(B) Spatial infinite volume and locality. Within the strong-coupling/KP corridor, we *do* construct the spatial thermodynamic limit $L \to \infty$: Section I builds an infinite-volume slab DLR state, the limiting boundary laws on finite windows, and the corresponding limiting transfer specification, and it shows that the Euclidean-time clustering bounds for local observables persist in this limit with constants uniform in L. Outside the KP corridor (or for a renormalised scaling limit leading to a full Euclidean theory on $\mathbb{R}^4$), spatial infinite-volume control remains a separate infrared problem.

(C) Full OS reconstruction and Euclidean invariance. OS reconstruction in the classical sense uses reflection positivity and invariance in *all* Euclidean directions (together with regularity). This paper performs reconstruction only in the Euclidean time direction and identifies the time translation operator. To obtain a complete relativistic QFT, one would typically need:

- a family of Schwinger functions satisfying the OS axioms on $\mathbb{R}^4$ (or a constructive substitute);
- sufficient regularity to reconstruct fields as operator-valued distributions;
- Euclidean invariance (rotations and translations) in the continuum limit.

The transfer-operator gap shown here is therefore best read as a *time-axis spectral statement* compatible with, and potentially feeding into, a full OS reconstruction if the additional axioms can be verified in a limiting theory.

(D) Interpreting m_* **as a physical mass gap.** Within the slab/transfer framework, the quantity $m_* = |\log \rho_*|/t$ (Section 14) is a Euclidean-time decay rate for correlations and a bound on the nontrivial spectrum of the transfer operator. In a full continuum theory, one usually identifies such a decay rate with a relativistic mass gap using spectral representations and covariance. Here we do not claim that identification beyond the transfer-operator context: the theorems guarantee a Euclidean-time spectral separation above the vacuum *within the reconstructed time-axis Hilbert space*. Whether this coincides with a relativistic mass gap in a continuum Minkowski theory depends on the success of the additional steps above.

14.11 Summary of scope

For ease of citation, we restate the scope of the main conclusions.

- At fixed regulator, the paper gives an explicit slab/transfer formulation of the Euclidean-time theory and proves: (i) reflection positivity in time, (ii) exponential clustering for gauge-invariant interior-supported bounded slab observables, and (iii) a transfer-operator spectral gap bound corresponding to the clustering rate (under a mild density hypothesis for Route A; Route B in the Wilson corridor uses the transfer L^2 gap directly).
- Under the uniformity assumptions made explicit in Section 14, these conclusions hold with constants independent of Reg.
- The paper does not construct a continuum limit on $\mathbb{R}^4$ and does not attempt a full OS reconstruction on $\mathbb{R}^4$. Spatial infinite volume is treated only within the strong-coupling/KP corridor: Section I constructs the thermodynamic limit $L \to \infty$ for the Wilson slab family under an L-uniform corridor hypothesis. Outside that corridor, results are stated at fixed L. Overall, the manuscript supplies a regulator-robust mechanism for Euclidean-time clustering and a time-axis transfer-operator gap within regulated Yang–Mills slab setups.

This completes the paper's main technical narrative: an auditable route from quantitative Harris mixing to regulator-uniform Euclidean-time clustering and a corresponding transfer-operator spectral gap bound for gauge-invariant observables.

A A Sobolev $H^1 \to L^4$ embedding on the slab

A.1 Statement and norms

Let $t > 0$ and $L > 0$, and write

$$S_{t,L} \coloneqq [0,t] \times \mathbb{T}^3_L, \qquad \mathbb{T}^3_L = (\mathbb{R}/L\mathbb{Z})^3,$$

equipped with the product flat metric and volume measure $\mathrm{d}X = \mathrm{d}x_0\, \mathrm{d}x$.

For a (scalar) function $u \in H^1(S_{t,L})$ define

$$\|u\|_{L^p(S_{t,L})} \coloneqq \Big(\int_{S_{t,L}} |u(X)|^p \,\mathrm{d}X\Big)^{1/p}, \qquad \|u\|^2_{H^1(S_{t,L})} \coloneqq \|u\|^2_{L^2(S_{t,L})} + \|\nabla u\|^2_{L^2(S_{t,L})},$$

where $\nabla = (\partial_0, \partial_{x_1}, \partial_{x_2}, \partial_{x_3})$. For $\mathbb{R}^m$-valued (or $\mathfrak{g}$-valued) functions, all norms are understood componentwise using the fixed inner product.

Proposition A.1 (Uniform Sobolev embedding on $S_{t,L}$)**.** *Fix $t > 0$ and assume $L \geq 1$. There exists a constant $C_S(t) < \infty$ (independent of L and independent of any UV truncation) such that for all $u \in H^1(S_{t,L})$,*

$$\|u\|_{L^4(S_{t,L})} \;\leq\; C_S(t)\, \|u\|_{H^1(S_{t,L})}. \tag{187}$$

Corollary A.2 (Dirichlet-in-time variant)**.** *Fix $t > 0$ and $L \geq 1$. There exists $C'_S(t) < \infty$ independent of L such that for all $u \in H^1(S_{t,L})$ whose trace satisfies $u(t, \cdot) = 0$ on $\mathbb{T}^3_L$ (in the $H^{1/2}$ trace sense),*

$$\|u\|_{L^4(S_{t,L})} \;\leq\; C'_S(t)\, \|\nabla u\|_{L^2(S_{t,L})}. \tag{188}$$

A.2 A fixed partition of unity with uniform overlap

We construct a partition of unity by translates of a fixed bump function, with overlap bounds independent of L (as long as $L \geq 1$).

Lemma A.3 (One-dimensional bump partition)**.** *There exists $\psi \in C_c^\infty((-2,2))$ with $0 \leq \psi \leq 1$ such that*

$$\sum_{m\in\mathbb{Z}} \psi(x-m) = 1 \qquad \text{for all } x \in \mathbb{R}, \tag{189}$$

and such that $\sup_{x\in\mathbb{R}} |\psi^{(j)}(x)| < \infty$ for $j = 0, 1$. Moreover, each $x \in \mathbb{R}$ belongs to the support of at most N_1 translates $\psi(\cdot - m)$, where N_1 is a fixed integer independent of x.

Proof. Choose any nonnegative $\varphi \in C_c^\infty((-1,1))$ with $\varphi > 0$ on $(-1,1)$. Define $Z(x) := \sum_{m\in\mathbb{Z}} \varphi(x-m)$, which is smooth, strictly positive, and 1-periodic. Set $\psi(x) := \varphi(x)/Z(x)$. Then $\psi \in C_c^\infty((-1,1)) \subset C_c^\infty((-2,2))$, $0 \leq \psi \leq 1$, and

$$\sum_{m\in\mathbb{Z}} \psi(x-m) = \frac{\sum_{m\in\mathbb{Z}} \varphi(x-m)}{Z(x)} = 1.$$

Uniform bounds on ψ and ψ' follow from smoothness and periodicity of Z and the compact support of φ. Finally, because ψ is supported in an interval of length < 4, at most $N_1 = 4$ translates can overlap at any point. □

We use Lemma A.3 in each coordinate.

Lemma A.4 (Product partition on the slab with uniform overlap)**.** *Fix $t > 0$ and $L \geq 1$. There exists a finite index set $\mathcal{I} = \mathcal{I}(t,L)$ and a family of functions $\{\eta_\alpha\}_{\alpha\in\mathcal{I}} \subset C^\infty([0,t] \times [0,L)^3)$ such that:*

(i) *$0 \leq \eta_\alpha \leq 1$ and $\sum_{\alpha\in\mathcal{I}} \eta_\alpha(X) = 1$ for all $X \in [0,t] \times [0,L)^3$;*

(ii) *each η_α is supported in a set of the form*

$$\operatorname{supp}(\eta_\alpha) \subset I_\alpha \times \prod_{i=1}^{3} J_{\alpha,i},$$

where $I_\alpha \subset \mathbb{R}$ has length < 4 and each $J_{\alpha,i} \subset \mathbb{R}$ has length < 4;

(iii) *there exists an integer N_* independent of (t,L) such that each point $X \in [0,t] \times [0,L)^3$ belongs to the supports of at most N_* of the functions η_α;*

(iv) *there exists $C_\eta < \infty$ independent of (t,L) such that*

$$\sup_{\alpha\in\mathcal{I}} \sup_{X\in[0,t]\times[0,L)^3} |\nabla \eta_\alpha(X)| \leq C_\eta. \tag{190}$$

Proof. Let ψ be as in Lemma A.3. On $\mathbb{R} \times \mathbb{R}^3$ define

$$\eta_{m_0,m_1,m_2,m_3}(x_0,x_1,x_2,x_3) := \psi(x_0-m_0)\,\psi(x_1-m_1)\,\psi(x_2-m_2)\,\psi(x_3-m_3), \qquad m \in \mathbb{Z}^4.$$

Then $\sum_{m\in\mathbb{Z}^4} \eta_m \equiv 1$ pointwise on $\mathbb{R}^4$ by the product structure.

We now restrict to the fundamental domain $[0,t] \times [0,L)^3$ representing $S_{t,L} = [0,t] \times \mathbb{T}_L^3$. Because ψ has compact support, only finitely many indices $m \in \mathbb{Z}^4$ have $\operatorname{supp}(\eta_m)$ intersecting $[0,t] \times [0,L)^3$.

Let $\mathcal{I}(t, L)$ be the set of such indices and define η_α to be the restriction of η_m to $[0, t] \times [0, L)^3$ for $\alpha = m \in \mathcal{I}(t, L)$. Since at each point in $[0, t] \times [0, L)^3$ only finitely many η_m are nonzero, the restricted family still satisfies $\sum_{\alpha \in \mathcal{I}} \eta_\alpha = 1$ pointwise on $[0, t] \times [0, L)^3$.

Properties (i)–(ii) are immediate from the construction. For (iii), note that in each coordinate, at most N_1 translates overlap at any point (Lemma A.3), so in 4 dimensions at most $N_* := N_1^4$ products can overlap at any point. For (iv), $\nabla \eta_m$ is a linear combination of products of ψ and ψ', hence uniformly bounded by $\sup |\psi|$ and $\sup |\psi'|$, independently of (t, L). □

A.3 Local Euclidean Sobolev on patches

We use the classical Sobolev embedding $H^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ for compactly supported functions.

Theorem A.5 (Sobolev on $\mathbb{R}^4$)**.** *There exists $C_{\mathbb{R}^4} < \infty$ such that for all $v \in H^1(\mathbb{R}^4)$ with compact support,*

$$\|v\|_{L^4(\mathbb{R}^4)} \leq C_{\mathbb{R}^4} \|v\|_{H^1(\mathbb{R}^4)}. \tag{191}$$

Proof. This is the classical Sobolev embedding in the critical case $n = 4$, $p = 4$. A proof can be found, for example, in standard texts on Sobolev spaces and elliptic PDE [1, 2]. □

We also need a simple product estimate for H^1 norms of cutoff functions.

Lemma A.6 (Cutoff product estimate)**.** *Let $\eta \in W^{1,\infty}(\mathbb{R}^4)$ and $u \in H^1(\mathbb{R}^4)$. Then $\eta u \in H^1(\mathbb{R}^4)$ and*

$$\|\eta u\|_{H^1(\mathbb{R}^4)} \leq \|\eta\|_{L^\infty} \|u\|_{H^1(\mathbb{R}^4)} + \|\nabla \eta\|_{L^\infty} \|u\|_{L^2(\mathbb{R}^4)}. \tag{192}$$

Proof. Since $\eta \in W^{1,\infty}$, multiplication by η maps L^2 to L^2 boundedly and maps ∇u to $\eta \nabla u$ in L^2. Moreover, $\nabla(\eta u) = (\nabla \eta) u + \eta \nabla u$ in the sense of distributions. Thus

$$\|\eta u\|_{L^2} \leq \|\eta\|_\infty \|u\|_{L^2}, \qquad \|\nabla(\eta u)\|_{L^2} \leq \|\nabla \eta\|_\infty \|u\|_{L^2} + \|\eta\|_\infty \|\nabla u\|_{L^2},$$

which gives (192). □

A.4 Proof of Proposition A.1

Proof of Proposition A.1. Let $\{\eta_\alpha\}_{\alpha \in \mathcal{I}}$ be the partition of unity from Lemma A.4. We identify $S_{t,L} = [0, t] \times \mathbb{T}_L^3$ with the fundamental domain $[0, t] \times [0, L)^3$ for the purpose of computing L^p and H^1 norms.

Let $E_{t,L} : H^1([0, t] \times [0, L)^3) \to H^1(\mathbb{R}^4)$ be a bounded extension operator for the rectangular box. Write

$$\|E_{t,L} f\|_{H^1(\mathbb{R}^4)} \leq C_{\mathrm{ext}}(t) \|f\|_{H^1([0,t] \times [0,L)^3)},$$

with $C_{\mathrm{ext}}(t) < \infty$ independent of L (and independent of any UV truncation).

For each α, apply Theorem A.5 to the compactly supported function $E_{t,L}(\eta_\alpha u)$:

$$\|\eta_\alpha u\|_{L^4(S_{t,L})} \leq \|E_{t,L}(\eta_\alpha u)\|_{L^4(\mathbb{R}^4)} \leq C_{\mathbb{R}^4} \|E_{t,L}(\eta_\alpha u)\|_{H^1(\mathbb{R}^4)} \leq C_{\mathbb{R}^4} C_{\mathrm{ext}}(t) \|\eta_\alpha u\|_{H^1(S_{t,L})}. \tag{193}$$

Step 1: reduce $\|u\|_{L^4}$ to patch norms. At each $X \in S_{t,L}$, at most N_* of the $\eta_\alpha(X)$ are nonzero (Lemma A.4(iii)). Thus, using $(\sum_{j=1}^m a_j)^4 \leq m^3 \sum_{j=1}^m a_j^4$ for nonnegative a_j,

$$|u(X)|^4 = \Big|\sum_{\alpha \in \mathcal{I}} \eta_\alpha(X) u(X)\Big|^4 \leq N_*^3 \sum_{\alpha \in \mathcal{I}} |\eta_\alpha(X) u(X)|^4. \tag{194}$$

Integrating (194) over $S_{t,L}$ gives

$$\|u\|^4_{L^4(S_{t,L})} \le N_*^3 \sum_{\alpha\in\mathcal{I}} \|\eta_\alpha u\|^4_{L^4(S_{t,L})}. \tag{195}$$

Step 2: apply Sobolev on each patch.

By (193),

$$\|\eta_\alpha u\|^4_{L^4(S_{t,L})} \le \big(C_{\mathbb{R}^4} C_{\mathrm{ext}}(t)\big)^4 \|\eta_\alpha u\|^4_{H^1(S_{t,L})}.$$

Insert this into (195):

$$\|u\|^4_{L^4(S_{t,L})} \le N_*^3 \big(C_{\mathbb{R}^4} C_{\mathrm{ext}}(t)\big)^4 \sum_{\alpha\in\mathcal{I}} \|\eta_\alpha u\|^4_{H^1(S_{t,L})}. \tag{196}$$

Step 3: replace the sum of fourth powers by the square of the sum of squares.

For nonnegative numbers a_α, $\sum_\alpha a_\alpha^2 \le (\sum_\alpha a_\alpha)^2$. Apply this with $a_\alpha = \|\eta_\alpha u\|^2_{H^1(S_{t,L})}$:

$$\sum_{\alpha\in\mathcal{I}} \|\eta_\alpha u\|^4_{H^1(S_{t,L})} \le \Big(\sum_{\alpha\in\mathcal{I}} \|\eta_\alpha u\|^2_{H^1(S_{t,L})}\Big)^2. \tag{197}$$

Step 4: bound $\sum_\alpha \|\eta_\alpha u\|^2_{H^1}$ by $\|u\|^2_{H^1}$. First, for the L^2 part:

$$\sum_{\alpha\in\mathcal{I}} \|\eta_\alpha u\|^2_{L^2(S_{t,L})} = \int_{S_{t,L}} \sum_{\alpha\in\mathcal{I}} \eta_\alpha(X)^2\, |u(X)|^2 \,\mathrm{d}X \le \int_{S_{t,L}} \Big(\sum_{\alpha\in\mathcal{I}} \eta_\alpha(X)\Big) |u(X)|^2 \,\mathrm{d}X = \|u\|^2_{L^2(S_{t,L})},$$

since $0 \le \eta_\alpha \le 1$ implies $\eta_\alpha^2 \le \eta_\alpha$ and $\sum_\alpha \eta_\alpha = 1$.

For the gradient part, use $\nabla(\eta_\alpha u) = \eta_\alpha \nabla u + (\nabla \eta_\alpha) u$ and $(a+b)^2 \le 2a^2 + 2b^2$:

$$\|\nabla(\eta_\alpha u)\|^2_{L^2} \le 2\|\eta_\alpha \nabla u\|^2_{L^2} + 2\|(\nabla\eta_\alpha) u\|^2_{L^2}.$$

Summing over α and using finite overlap gives

$$\sum_\alpha \|\eta_\alpha \nabla u\|^2_{L^2(S_{t,L})} = \int_{S_{t,L}} \sum_\alpha \eta_\alpha(X)^2\, |\nabla u(X)|^2 \,\mathrm{d}X \le \|\nabla u\|^2_{L^2(S_{t,L})},$$

and

$$\sum_\alpha \|(\nabla\eta_\alpha) u\|^2_{L^2(S_{t,L})} \le C_\eta^2\, N_*\, \|u\|^2_{L^2(S_{t,L})},$$

using Lemma A.4(iii) and (190). Therefore,

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}} \|\eta_\alpha u\|^2_{H^1(S_{t,L})} &\le \|u\|^2_{L^2} + 2\|\nabla u\|^2_{L^2} + 2C_\eta^2 N_*\, \|u\|^2_{L^2} \\ &\le C_{\mathrm{PU}}\, \|u\|^2_{H^1(S_{t,L})},\end{aligned} \tag{198}$$

where $C_{\mathrm{PU}} := \max\{1 + 2C_\eta^2 N_*, 2\}$ depends only on the fixed bump ψ (hence universal).

Step 5: conclude.

Combine (196), (197), and (198):

$$\|u\|^4_{L^4(S_{t,L})} \le N_*^3 \big(C_{\mathbb{R}^4} C_{\mathrm{ext}}(t)\big)^4 C_{\mathrm{PU}}^2\, \|u\|^4_{H^1(S_{t,L})}.$$

Taking fourth roots yields (187) with

$$C_S(t) := N_*^{3/4}\, C_{\mathbb{R}^4} C_{\mathrm{ext}}(t)\, C_{\mathrm{PU}}^{1/2}.$$

This constant is independent of L and independent of any UV truncation. □

A.5 Proof of Corollary A.2

We use a one-dimensional Poincaré inequality in the time direction.

Lemma A.7 (Poincaré inequality in time)**.** *Let $u \in H^1(S_{t,L})$ have trace $u(t,\cdot) = 0$ on $\mathbb{T}^3_L$. Then*

$$\|u\|_{L^2(S_{t,L})} \;\le\; t\,\|\partial_0 u\|_{L^2(S_{t,L})} \;\le\; t\,\|\nabla u\|_{L^2(S_{t,L})}. \tag{199}$$

Proof. By the trace theorem, the condition $u(t,\cdot) = 0$ implies that for almost every $x \in \mathbb{T}^3_L$, the function $x_0 \mapsto u(x_0, x)$ lies in $H^1(0,t)$ and satisfies $u(t,x) = 0$. Hence for a.e. (x_0, x),

$$u(x_0, x) = -\int_{x_0}^{t} \partial_0 u(s,x)\,\mathrm{d}s.$$

By Cauchy–Schwarz,

$$|u(x_0,x)|^2 \le (t - x_0)\int_{x_0}^{t} |\partial_0 u(s,x)|^2\,\mathrm{d}s \le t\int_0^t |\partial_0 u(s,x)|^2\,\mathrm{d}s.$$

Integrate over $x_0 \in [0,t]$ and $x \in \mathbb{T}^3_L$:

$$\int_{S_{t,L}} |u|^2\,\mathrm{d}X \le t^2 \int_{S_{t,L}} |\partial_0 u|^2\,\mathrm{d}X,$$

which is (199). The final inequality follows from $|\partial_0 u| \le |\nabla u|$ pointwise. □

Proof of Corollary A.2. Apply Proposition A.1 and then Lemma A.7:

$$\|u\|_{L^4(S_{t,L})} \le C_S(t)\big(\|u\|_{L^2(S_{t,L})} + \|\nabla u\|_{L^2(S_{t,L})}\big) \le C_S(t)(1+t)\,\|\nabla u\|_{L^2(S_{t,L})}.$$

Thus (188) holds with $C'_S(t) = C_S(t)(1+t)$. □

B Disintegration, conditional expectations, and monotone-class tools

This appendix records measure-theoretic lemmas used (sometimes implicitly) in Sections 11–13. All objects in the main text live on finite-dimensional Euclidean spaces with their Borel σ-algebras, hence are *standard Borel* spaces; in particular, regular conditional probabilities and disintegrations exist.

B.1 Standard Borel spaces and kernels

A *standard Borel space* is a measurable space isomorphic (as a measurable space) to a Borel subset of a Polish space. Every finite-dimensional Euclidean space with its Borel σ-algebra is standard Borel, and finite products of standard Borel spaces are standard Borel.

A *(Markov) kernel* from $(\mathsf{X}, \mathcal{B}_{\mathsf{X}})$ to $(\mathsf{Y}, \mathcal{B}_{\mathsf{Y}})$ is a map $K : \mathsf{X} \times \mathcal{B}_{\mathsf{Y}} \to [0,1]$ such that: (i) for each $x \in \mathsf{X}$, $A \mapsto K(x,A)$ is a probability measure on Y; (ii) for each $A \in \mathcal{B}_{\mathsf{Y}}$, $x \mapsto K(x,A)$ is $\mathcal{B}_{\mathsf{X}}$-measurable. We write $K(x,\mathrm{d}y)$ for the associated probability measure.

B.2 Disintegration / regular conditional probabilities

Lemma B.1 (Disintegration of a finite measure)**.** *Let* $(\mathsf{X}, \mathcal{B}_\mathsf{X})$ *and* $(\mathsf{Y}, \mathcal{B}_\mathsf{Y})$ *be standard Borel spaces, and let* μ *be a finite measure on* $\mathsf{X} \times \mathsf{Y}$*. Let* μ_X *denote the* X*-marginal of* μ*. Then there exists a kernel* $K(x, \mathrm{d}y)$ *on* Y *such that*

$$\mu(\mathrm{d}x, \mathrm{d}y) \;=\; \mu_\mathsf{X}(\mathrm{d}x)\, K(x, \mathrm{d}y), \tag{200}$$

in the sense that for every bounded measurable $\varphi : \mathsf{X} \times \mathsf{Y} \to \mathbb{R}$*,*

$$\int_{\mathsf{X}\times\mathsf{Y}} \varphi(x, y)\, \mu(\mathrm{d}x, \mathrm{d}y) \;=\; \int_\mathsf{X} \Big(\int_\mathsf{Y} \varphi(x, y)\, K(x, \mathrm{d}y) \Big)\, \mu_\mathsf{X}(\mathrm{d}x). \tag{201}$$

Moreover, K *is* μ_X*-a.s. unique: if* K' *also satisfies* (201)*, then* $K(x, \cdot) = K'(x, \cdot)$ *for* μ_X*-a.e.* x*.*

Proof. This is the standard disintegration theorem / existence of regular conditional probabilities on standard Borel spaces. Since all spaces in the paper are finite-dimensional Euclidean (hence standard Borel), the hypotheses are met.

For references, see e.g. Bogachev [23] or Kallenberg [24]. Uniqueness μ_X-a.s. follows by testing against indicator functions of sets in $\mathcal{B}_\mathsf{Y}$ and using a π-λ argument. □

Remark B.2 (Interpretation as a conditional law)*.* If (X, Y) is an $\mathsf{X} \times \mathsf{Y}$-valued random element with law $\mu/\mu(\mathsf{X} \times \mathsf{Y})$, then $K(x, \cdot)$ can be taken as a version of the conditional law of Y given $X = x$, and (201) is precisely the tower/conditioning identity.

B.3 Iterated conditioning for Markov kernels

The next lemma is a convenient encapsulation of iterated conditioning for a Markov chain, used repeatedly when expressing stacked expectations as repeated integrals against the transfer kernel.

Lemma B.3 (Kernel iteration and conditional expectation)**.** *Let* K *be a Markov kernel on a standard Borel space* H *and let* $(B_n)_{n\ge 0}$ *be a Markov chain with transition kernel* K*. For any bounded measurable* $g : \mathsf{H} \to \mathbb{C}$ *and any* $n \in \mathbb{N}$*,*

$$\mathbb{E}[g(B_n) \,|\, B_0] \;=\; (\mathsf{T}^n g)(B_0), \qquad (\mathsf{T}g)(b) := \int_\mathsf{H} g(b')\, \mathsf{K}(b, \mathrm{d}b'). \tag{202}$$

If π *is an invariant probability for* K *and* $B_0 \sim \pi$*, then for all* $f, g \in L^2(\pi)$*,*

$$\mathbb{E}_\pi\big[\overline{f(B_0)}\, g(B_n)\big] \;=\; \langle f, \mathsf{T}^n g\rangle_{L^2(\pi)}. \tag{203}$$

Proof. For $n = 1$, (202) is the definition of conditional expectation under the Markov property: $\mathbb{E}[g(B_1) \,|\, B_0 = b] = \int g(b')\mathsf{K}(b, \mathrm{d}b') = (\mathsf{T}g)(b)$. Assume (202) holds for n. Then by the tower property and the Markov property,

$$\mathbb{E}[g(B_{n+1}) \,|\, B_0] = \mathbb{E}\big[\,\mathbb{E}[g(B_{n+1}) \,|\, B_1] \,\big|\, B_0\big] = \mathbb{E}\big[(\mathsf{T}^n g)(B_1) \,\big|\, B_0\big] = (\mathsf{T}(\mathsf{T}^n g))(B_0) = (\mathsf{T}^{n+1} g)(B_0),$$

proving (202) for all n by induction. If $B_0 \sim \pi$, then

$$\mathbb{E}_\pi[\overline{f(B_0)}\, g(B_n)] = \mathbb{E}_\pi\big[\overline{f(B_0)}\, \mathbb{E}[g(B_n) \,|\, B_0]\big] = \mathbb{E}_\pi\big[\overline{f(B_0)}\, (\mathsf{T}^n g)(B_0)\big] = \langle f, \mathsf{T}^n g\rangle_{L^2(\pi)},$$

which is (203). □

B.4 Monotone class: extending identities from factorised cylinders

We use the following functional monotone class lemma to justify proofs that first treat factorised cylinder functions and then extend to all bounded measurable functions.

Lemma B.4 (Functional monotone class lemma)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $\mathcal{A} \subset L^\infty(\Omega, \mathcal{F}; \mathbb{R})$ be an* algebra *of bounded measurable functions (i.e. closed under pointwise products and linear combinations) that generates $\mathcal{F}$ as a σ-algebra:*

$$\sigma(\mathcal{A}) \;=\; \mathcal{F}.$$

Let $\mathcal{M} \subset L^\infty(\Omega, \mathcal{F}; \mathbb{R})$ be a vector space of bounded measurable functions such that:

(i) $\mathcal{A} \subset \mathcal{M}$;

(ii) *$\mathcal{M}$ is closed under bounded monotone limits: if $0 \le f_n \uparrow f$ pointwise with $\sup_n \|f_n\|_\infty < \infty$ and each $f_n \in \mathcal{M}$, then $f \in \mathcal{M}$.*

Then $\mathcal{M} = L^\infty(\Omega, \mathcal{F}; \mathbb{R})$.

Proof. Define
$$\mathcal{C} := \{\, A \in \mathcal{F} : \mathbf{1}_A \in \mathcal{M} \,\}.$$
We claim that $\mathcal{C}$ is a *Dynkin system* (a λ-system):

(a) $\Omega \in \mathcal{C}$ because $\mathbf{1}_\Omega = 1 \in \mathcal{A} \subset \mathcal{M}$.

(b) If $A \in \mathcal{C}$, then $A^c \in \mathcal{C}$ because $\mathbf{1}_{A^c} = 1 - \mathbf{1}_A$ and $\mathcal{M}$ is a vector space.

(c) If $A_1, A_2, \dots \in \mathcal{C}$ are disjoint, then $\bigcup_n A_n \in \mathcal{C}$ because the partial sums $s_N = \sum_{n=1}^N \mathbf{1}_{A_n}$ belong to $\mathcal{M}$ and satisfy $0 \le s_N \uparrow \mathbf{1}_{\cup_n A_n}$, so closure under bounded monotone limits gives $\mathbf{1}_{\cup_n A_n} \in \mathcal{M}$.

Hence $\mathcal{C}$ is a λ-system.

Next, let
$$\mathcal{P} := \{\, A \in \mathcal{F} : \mathbf{1}_A \in \mathcal{A} \,\}.$$
Because $\mathcal{A}$ is an algebra, $\mathcal{P}$ is a π-system: if $A, B \in \mathcal{P}$, then $\mathbf{1}_{A \cap B} = \mathbf{1}_A \mathbf{1}_B \in \mathcal{A}$, so $A \cap B \in \mathcal{P}$. Also $\sigma(\mathcal{P}) = \sigma(\mathcal{A}) = \mathcal{F}$ by assumption.

Moreover, $\mathcal{P} \subset \mathcal{C}$ because $\mathcal{A} \subset \mathcal{M}$ implies $\mathbf{1}_A \in \mathcal{M}$. By the π-λ theorem, $\mathcal{C}$ contains $\sigma(\mathcal{P}) = \mathcal{F}$, hence $\mathbf{1}_A \in \mathcal{M}$ for all $A \in \mathcal{F}$.

Finally, every bounded measurable function f can be uniformly approximated by finite linear combinations of indicators of measurable sets (simple functions). Since $\mathcal{M}$ is a vector space containing all indicators and is closed under bounded monotone limits, it contains all bounded measurable functions (write $f = f^+ - f^-$ and approximate $f^\pm$ by increasing simple functions). Thus $\mathcal{M} = L^\infty(\Omega, \mathcal{F}; \mathbb{R})$. □

B.5 Conditional independence via factorisation of conditional expectations

The following lemma provides a concise criterion for conditional independence that we use when splitting the stacked field into past and future halves given the boundary trace at time 0.

Lemma B.5 (Factorisation criterion for conditional independence)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Let $X_1, \ldots, X_n$ be random elements taking values in standard Borel spaces. Assume that for every choice of bounded measurable functions f_k,*

$$\mathbb{E}\Big[\prod_{k=1}^{n} f_k(X_k)\,\Big|\,\mathcal{G}\Big] = \prod_{k=1}^{n} \mathbb{E}\big[f_k(X_k)\,|\,\mathcal{G}\big] \qquad \mathbb{P}\text{-}a.s. \tag{204}$$

Then $(X_1, \ldots, X_n)$ are conditionally independent given $\mathcal{G}$, i.e. for all measurable sets A_k,

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n\,|\,\mathcal{G}) = \prod_{k=1}^{n} \mathbb{P}(X_k \in A_k\,|\,\mathcal{G}) \qquad \mathbb{P}\text{-}a.s. \tag{205}$$

Proof. Fix measurable sets A_k and set $f_k = \mathbf{1}_{A_k}$ in (204). Then (204) becomes exactly (205). Thus the joint conditional law factors on rectangles. Because the X_k take values in standard Borel spaces, the collection of measurable rectangles $\{A_1 \times \cdots \times A_n\}$ is a π-system generating the product σ-algebra. A standard π-λ argument extends the factorisation from rectangles to the whole product σ-algebra, yielding conditional independence. $\square$

C Gauge-invariant observables: Wilson loops and bounded curvature functionals

This appendix records self-contained proofs of the elementary properties used in Section 12: boundedness and gauge invariance of the example observables. All statements hold at fixed regulator, where fields are smooth enough for the ODE definition of holonomy.

C.1 Holonomy transforms by conjugation

Lemma C.1 (Gauge transformation of holonomy)**.** *Let γ be a piecewise C^1 path and let A be a (regulated) connection one-form. For a gauge transform g, the parallel transport satisfies*

$$U_{\gamma,A^g}(s) = g(\gamma(s))^{-1}\,U_{\gamma,A}(s)\,g(\gamma(0)).$$

In particular, for a closed loop $\gamma(1) = \gamma(0)$,

$$\mathrm{Hol}_\gamma(A^g) = g(\gamma(0))^{-1}\,\mathrm{Hol}_\gamma(A)\,g(\gamma(0)).$$

Proof. Differentiate the candidate right-hand side and use the defining ODE for $U_{\gamma,A}$ and the transformation rule $A^g = \mathrm{Ad}_{g^{-1}}A + g^{-1}\,\mathrm{d}g$. Uniqueness of solutions to linear ODEs implies the identity. $\square$

C.2 Wilson loops are bounded and gauge-invariant

Lemma C.2 (Boundedness and gauge invariance of Wilson loops)**.** *For a unitary representation $\varrho : G \to U(d_\varrho)$ and a closed loop γ, the Wilson loop (see e.g. [46, 49, 50, 52])*

$$\mathcal{W}_{\gamma,\varrho}(A) = \frac{1}{d_\varrho}\,\mathrm{Tr}\,\varrho(\mathrm{Hol}_\gamma(A))$$

satisfies $|\mathcal{W}_{\gamma,\varrho}(A)| \le 1$ for all A and is gauge-invariant.

Proof. Boundedness follows from $|\mathrm{Tr}(U)| \le d_\varrho$ for unitary U. Gauge invariance follows from Lemma C.1 and the conjugation invariance of the trace. $\square$

C.3 Curvature-energy observables

Lemma C.3 (Gauge invariance of $\int \phi \langle F, F\rangle$)**.** *Let ϕ be a scalar test function and define*

$$\mathcal{E}_\phi(A) = \int \phi(x) \, \langle F_{\mu\nu}(A)(x), F_{\mu\nu}(A)(x)\rangle_{\mathfrak{g}} \, \mathrm{d}x.$$

Then $\mathcal{E}_\phi(A^g) = \mathcal{E}_\phi(A)$ for all gauge transforms g. Consequently, for bounded measurable Ψ, the observable $\Psi(\mathcal{E}_\phi(A))$ is gauge-invariant and bounded.

Proof. Use $F(A^g) = \mathrm{Ad}_{g^{-1}} F(A)$ and Ad-invariance of the inner product on $\mathfrak{g}$. $\square$

D A spectral lemma: exponential correlation decay implies spectral support

This appendix isolates the spectral-measure argument used in Section 13 to convert exponential decay of even-time correlations into a spectral inclusion for a self-adjoint contraction (cf. [42]).

Lemma D.1 (Decay of even powers forces spectral support)**.** *Let T be a self-adjoint contraction on a Hilbert space $\mathcal{H}$, with spectral resolution $\mathsf{E}(\cdot)$. For $f \in \mathcal{H}$, define $\mu_f(A) = \langle f, \mathsf{E}(A) f\rangle$. Assume there exist $\rho \in (0,1)$ and $C_f < \infty$ such that*

$$\langle f, \mathsf{T}^{2n} f\rangle \le C_f \, \rho^{2n} \qquad \text{for all } n \in \mathbb{N}.$$

Then μ_f is supported in $[-\rho, \rho]$.

Proof. If $\mu_f(\{|\lambda| > \rho\}) > 0$, then for some $\varepsilon > 0$, $\mu_f(\{|\lambda| \ge \rho + \varepsilon\}) > 0$. Using the spectral representation $\langle f, \mathsf{T}^{2n} f\rangle = \int \lambda^{2n} \, \mu_f(\mathrm{d}\lambda)$ gives the lower bound

$$\langle f, \mathsf{T}^{2n} f\rangle \ge (\rho + \varepsilon)^{2n} \, \mu_f(\{|\lambda| \ge \rho + \varepsilon\}),$$

which contradicts the assumed upper bound after dividing by ρ^{2n} and letting $n \to \infty$. $\square$

E Regulator-uniform weak Harris verification for the sampler

This appendix supplies the missing *regulator-uniform* inputs for Theorem 2.16. The point is to avoid the dimension blow-up coming from the naive "$\mathrm{Tr}(I) = d$" term in the finite-dimensional Itô computation for $\|B_s\|^2$.

We work in the abstract Wiener realisation $(E, \mathcal{H}, \mu_{0,t,\mathrm{Reg}})$ fixed in Section 2.4, but the quantitative estimates are expressed on the Cameron–Martin Hilbert space $\mathcal{H}$: the low-mode projection is taken in $\mathcal{H}$, and the noise is realised so that the quadratic variation contributing to Itô's formula is governed by a *trace-class* covariance with trace bounded uniformly in Reg.

E.1 Trace-class realisation of the cylindrical noise

Let $i_{t,\mathrm{Reg}} : \mathcal{H} \hookrightarrow E$ be the abstract Wiener injection. In this appendix we work in a *Hilbert* realisation of the ambient space E (e.g. a negative Sobolev space), so that the injection admits a Hilbert adjoint $i^*_{t,\mathrm{Reg}} : E \to \mathcal{H}$. Define the (trace-class) covariance operator on $\mathcal{H}$

$$Q^{\mathcal{H}}_{t,\mathrm{Reg}} \; := \; i^*_{t,\mathrm{Reg}} i_{t,\mathrm{Reg}} : \mathcal{H} \to \mathcal{H}.$$

Let $(W_s)_{s \ge 0}$ be an $\mathcal{H}$-valued $Q^{\mathcal{H}}_{t,\mathrm{Reg}}$–Wiener process in the sense of [19]. All Itô computations below are performed at the Galerkin level in $\mathcal{H}$, where the quadratic variation is governed by $Q^{\mathcal{H}}_{t,\mathrm{Reg}}$.

Lemma E.1 (Uniform trace bound)**.** *There exists $\mathcal{T}_E(t) \in (0,\infty)$, depending on t (and on the chosen abstract Wiener realisation) but independent of* Reg*, such that*

$$\mathrm{Tr}(Q^{\mathcal{H}}_{t,\mathrm{Reg}}) = \|i_{t,\mathrm{Reg}}\|^2_{\mathrm{HS}} \le \mathcal{T}_E(t) \qquad \textit{for all } \mathrm{Reg}.$$

Proof. Let $(e_k)_{k\ge1}$ be an orthonormal basis of $\mathcal{H}$. Since $Q^{\mathcal{H}}_{t,\mathrm{Reg}} = i^*_{t,\mathrm{Reg}} i_{t,\mathrm{Reg}}$,

$$\mathrm{Tr}(Q^{\mathcal{H}}_{t,\mathrm{Reg}}) = \sum_{k\ge1} \left\langle Q^{\mathcal{H}}_{t,\mathrm{Reg}} e_k, e_k \right\rangle_{\mathcal{H}} = \sum_{k\ge1} \langle i_{t,\mathrm{Reg}} e_k, i_{t,\mathrm{Reg}} e_k\rangle_E = \sum_{k\ge1} \|i_{t,\mathrm{Reg}} e_k\|^2_E = \|i_{t,\mathrm{Reg}}\|^2_{\mathrm{HS}}.$$

By hypothesis on the chosen Hilbert realisation E, $\sup_{\mathrm{Reg}} \|i_{t,\mathrm{Reg}}\|^2_{\mathrm{HS}} < \infty$ at fixed t; denote this supremum by $\mathcal{T}_E(t)$. ☐

Remark E.2 (Concrete choice of E and the trace bound $\mathcal{T}_E(t)$)**.** Appendix E only needs a Hilbert realisation of E for which the abstract Wiener injection $i_{t,\mathrm{Reg}} : \mathcal{H} \hookrightarrow E$ is Hilbert–Schmidt *with a bound uniform in* Reg. In concrete regulator schemes one can arrange this by working in a sufficiently negative Sobolev topology and, if needed, including the natural spatial-volume normalisation in the ambient norm. For instance, one may take $E = H^{-r}(\mathbb{T}^3_L)$ with $r > 3/2$ and equip E with an L-dependent equivalent Hilbert norm such that

$$\|i_{t,\mathrm{Reg}}\|^2_{\mathrm{HS}} \;=\; \mathrm{Tr}(Q^{\mathcal{H}}_{t,\mathrm{Reg}}) \;=\; L^{-3} \sum_{k\in(2\pi/L)\mathbb{Z}^3} (1+|k|^2)^{-r},$$

which is finite for $r > 3/2$ and bounded uniformly in $L \ge 1$; any additional UV cutoff only reduces the sum. Other negative-regularity choices lead to analogous uniform trace bounds.

E.2 Uniform drift (Lyapunov) bound

Write the sampler as the E-valued diffusion

$$dB_s = -\big(B_s + \nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}(B_s)\big)\, ds + \sqrt{2}\, dW_s,$$

where W_s is the realised Q–Wiener process and $\nabla_{\mathcal{H}} U$ denotes the Cameron–Martin gradient. Let $V : \mathcal{H} \to [1,\infty)$ be the quadratic Lyapunov function

$$V(x) \;\coloneqq\; 1 + \|x\|^2_{\mathcal{H}}.$$

The key input is the regulator-uniform one-sided growth bound (proved in the main body as part of the structural package):

Lemma E.3 (Uniform one-sided growth)**.** *There exist constants $K_1(t) \in (0,\infty)$ and $K_0(t) \in [0,\infty)$, independent of* Reg*, such that for all $x \in \mathcal{H}$,*

$$\big|\langle x, \nabla_{\mathcal{H}} U_{t,\mathrm{Reg}}(x)\rangle_{\mathcal{H}}\big| \;\le\; K_1(t)\, \|x\|_{\mathcal{H}} + K_0(t).$$

Proof. This is exactly Lemma 6.9 in Section 6. ☐

Lemma E.4 (Regulator-uniform Lyapunov drift)**.** *Fix $t > 0$ and $\tau > 0$. There exist constants $\lambda \in (0,1)$ and $K < \infty$, depending on (t,τ) but independent of* Reg*, such that*

$$(P_\tau V)(x) \;\le\; \lambda\, V(x) + K, \qquad x \in \mathcal{H}.$$

Proof. Let $(B_s)_{s\geq 0}$ solve the sampler with $B_0 = x$. In the present infinite-dimensional setting, this identity is understood by Galerkin approximation (cf. e.g. [19, Ch. 4]): let Π_N be the orthogonal projection onto an N-dimensional subspace of $\mathcal{H}$ containing $\Pi_m\mathcal{H}$, and let $B_s^{(N)} := \Pi_N B_s$ solve the projected SDE driven by $\Pi_N W_s$. Applying the standard finite-dimensional Itô formula to $\|B_s^{(N)}\|_{\mathcal{H}}^2$ produces the quadratic-variation term $2\,\mathrm{Tr}(\Pi_N Q_{t,\mathrm{Reg}}^{\mathcal{H}}\Pi_N)$, which is monotone in N and bounded by $2\,\mathrm{Tr}(Q_{t,\mathrm{Reg}}^{\mathcal{H}})$. Letting $N\to\infty$ is justified as follows: the projected solutions $B^{(N)}$ converge to B in $L^2(\Omega; C([0,T];\mathcal{H}))$ for each $T<\infty$ by standard stability of finite-dimensional approximations of globally well-posed SDEs. Moreover, since $Q_{t,\mathrm{Reg}}^{\mathcal{H}}$ is trace-class and $\mathrm{Tr}(\Pi_N Q_{t,\mathrm{Reg}}^{\mathcal{H}}\Pi_N)\uparrow \mathrm{Tr}(Q_{t,\mathrm{Reg}}^{\mathcal{H}})$, monotone convergence yields the quadratic-variation limit. Together with the Lyapunov bound controlling the drift terms, one may pass to the limit in the Itô identity in expectation, yielding the displayed formula with no appearance of $\dim(\mathcal{H})$.

$$\frac{d}{ds}\mathbb{E}\|B_s\|_{\mathcal{H}}^2 = -2\,\mathbb{E}\|B_s\|_{\mathcal{H}}^2 - 2\,\mathbb{E}\langle B_s, \nabla_{\mathcal{H}}U(B_s)\rangle_{\mathcal{H}} + 2\,\mathrm{Tr}(Q_{t,\mathrm{Reg}}^{\mathcal{H}}).$$

Apply Lemma E.3 and Lemma E.1, then Young's inequality $2K_1\|x\|\leq \|x\|^2 + K_1^2$, to obtain

$$\frac{d}{ds}\mathbb{E}\|B_s\|_{\mathcal{H}}^2 \ \leq\ -\mathbb{E}\|B_s\|_{\mathcal{H}}^2 + \big(K_1(t)^2 + 2K_0(t) + 2\mathcal{T}_E(t)\big)\ =:\ C(t).$$

Gronwall gives

$$\mathbb{E}\|B_\tau\|_{\mathcal{H}}^2 \leq e^{-\tau}\|x\|_{\mathcal{H}}^2 + C(t)\,(1-e^{-\tau}),$$

hence

$$(P_\tau V)(x) = 1 + \mathbb{E}\|B_\tau\|_{\mathcal{H}}^2 \leq e^{-\tau}(1+\|x\|_{\mathcal{H}}^2) + (1-e^{-\tau})(1+C(t)).$$

This is the claimed drift inequality with $\lambda = e^{-\tau}$ and $K = (1-e^{-\tau})(1+C(t))$, uniformly in Reg. $\square$

E.3 Projected minorisation (full Girsanov + Gaussian lower bound)

Fix a finite-rank orthogonal projection $\Pi_m : \mathcal{H}\to\mathcal{H}$ of rank m, chosen *independently of* Reg (e.g. the first m spatial Fourier modes once Λ exceeds the corresponding threshold). Let $B_R := \{x\in\mathcal{H} : \|x\|_{\mathcal{H}}\leq R\}$.

Lemma E.5 (Projected minorisation on B_R)**.** *Fix $R>0$. There exist a time $s_\star\in(0,1]$, a radius $R_0\geq R$, a constant $\varepsilon\in(0,1]$, and a probability measure $\nu^\star$ on $\Pi_m\mathcal{H}$, all depending only on (t,R,m) but independent of* Reg*, such that for all $x\in B_R$,*

$$(\Pi_m)_\# P_{s_\star}(x,\cdot)\ \geq\ \varepsilon\,\nu^\star(\cdot).$$

Proof. We follow the standard three-step scheme: (i) compare the projected law to a projected OU law by Girsanov with uniform Novikov, (ii) lower bound the projected OU density uniformly on a ball, (iii) combine.

Step 1: uniform Girsanov lower bound on $\Pi_m\mathcal{H}$. Let $Z_s := \Pi_m B_s$ and let Y_s solve the m-dimensional OU SDE on $\Pi_m\mathcal{H}$,

$$dY_s = -Y_s\,ds + \sqrt{2}\,dW_s^{(m)}, \qquad Y_0 = \Pi_m x,$$

with $W^{(m)}$ an m-dimensional Brownian motion. Over the time interval $[0,s_\star]$ and for initial conditions $x\in B_R$, the projected drift difference is bounded in $L^2([0,s_\star])$ uniformly in Reg because $\nabla_{\mathcal{H}}U$ is Lipschitz on B_R (Theorem 2.15) and Π_m has fixed finite rank. Choosing $s_\star$ sufficiently small (depending only on (t,R,m)), Novikov's condition holds uniformly and Girsanov's theorem gives mutual absolute continuity of the laws of $Z_{s_\star}$ and $Y_{s_\star}$ on $\Pi_m\mathcal{H}$ with a Radon–Nikodým derivative bounded below by $\exp\{-C(t,R)\}$ for some $C(t,R)<\infty$ independent of Reg.

Step 2: uniform Gaussian density lower bound. The law of $Y_{s_\star}$ is the Gaussian $N(m_x, \Sigma_{OU}(s_\star))$ with

$$m_x = e^{-s_\star}\Pi_m x, \qquad \Sigma_{OU}(s_\star) = \int_0^{s_\star} 2e^{-2u}\,du\; I_{\Pi_m\mathcal{H}} = (1-e^{-2s_\star})\, I_{\Pi_m\mathcal{H}}.$$

Fix $R_0 := R+1$ and define

$$\beta_*(t,m,R,s_\star) \;:=\; \inf_{x\in B_R}\ \inf_{z\in B_{R_0}\cap\Pi_m\mathcal{H}} \ \mathrm{density}_{Y_{s_\star}}(z) \;>\; 0,$$

which is strictly positive since m_x ranges over a compact set and $\Sigma_{OU}(s_\star)$ is nondegenerate on the fixed finite-dimensional space $\Pi_m\mathcal{H}$. Let $\nu^\star$ be the normalised restriction of $N(0,\Sigma_{OU}(s_\star))$ to $B_{R_0}\cap\Pi_m\mathcal{H}$. Then $P(Y_{s_\star}\in\cdot)\ge\beta_*\,\nu^\star(\cdot)$.

Step 3: conclusion. By the Girsanov lower bound from Step 1,

$$(\Pi_m)_\# P_{s_\star}(x,\cdot) \;\ge\; e^{-C(t,R)}\,(\Pi_m)_\# P^{OU}_{s_\star}(x,\cdot) \;\ge\; e^{-C(t,R)}\,\beta_*(t,m,R,s_\star)\,\nu^\star(\cdot).$$

Set $\varepsilon := e^{-C(t,R)}\beta_*(t,m,R,s_\star)$ to conclude. All constants are independent of Reg. □

E.4 Conclusion of Theorem 2.16

Proof of Theorem 2.16. Apply the weak Harris theorem (Theorem 7.11) to the one-step kernel $P_{s_\star}$, with drift from Lemma E.4 (applied at $\tau = s_\star$) and d-small set provided by Lemma E.5 (using the bounded cost $d_{m,\eta}$ from Section 2 which controls Π_m-displacement). This yields the geometric contraction (8) and hence (9), with constants independent of Reg. □

F A finite-range lattice verification of the transfer-kernel contraction

This appendix verifies Assumption 2.18 in a concrete regulator family, without appealing to any external large-scale construction. *Scope and uniformity.* Throughout this appendix, $L>0$ and $t>0$ are fixed. The regulator tuple varies only through the ultraviolet refinement (e.g. lattice spacing $a\downarrow 0$ or cutoff $\Lambda\uparrow\infty$) and auxiliary finite-dimensional truncations, while the *local* interaction structure is held fixed (Wilson-type plaquette terms with bounded range). The window W used for the Doeblin minorisation is fixed in *unit lattice/block scale* (so the number of degrees of freedom in W is uniformly bounded), and "uniform in Reg" below refers to uniformity in this UV/refinement parameter at fixed (t,L).

The point is simple: for *finite-range* Euclidean-time updates (such as the time-layer conditional resampling induced by a finite-range lattice slab specification), a fixed finite block of boundary degrees of freedom has a conditional density that is uniformly bounded below. This gives a Doeblin minorisation and hence geometric contraction for the associated transfer kernel.

F.1 A local Doeblin minorisation on a fixed window

Fix a finite set W of boundary degrees of freedom (e.g. a finite set of spatial links on a single time-slice), and let Π_W denote the corresponding coordinate projection.

Lemma F.1 (Uniform local Doeblin lower bound on a fixed window)**.** *Consider a Wilson-type finite-range lattice regulator family on the slab $S_{t,L}$ with compact gauge group G and ultraviolet refinement (e.g. lattice spacing $a \downarrow 0$) at fixed (t, L). Let W be a fixed finite set of boundary degrees of freedom (fixed in lattice units across refinements), let Π_W be the coordinate projection, and let Haar_W denote the normalised product Haar measure on G^W. Then there exists $\varepsilon_W = \varepsilon_W(\beta, t, L, G) \in (0, 1)$, independent of the UV refinement and auxiliary truncations, such that for all boundary configurations b,*

$$(\Pi_W)_\# K_{t,\mathrm{Reg}}(b, \cdot) \ \geq \ \varepsilon_W \, \mathrm{Haar}_W(\cdot). \tag{206}$$

Moreover one may take

$$\varepsilon_W \ = \ \exp\{-2M_W(\beta)\}, \qquad M_W(\beta) \ \leq \ N_W \, \beta \, \|\phi\|_\infty, \tag{207}$$

where ϕ is the single-plaquette potential (including any finite-range local counterterms), $\|\phi\|_\infty < \infty$ is its global supremum on G, and $N_W < \infty$ is the number of local action terms touching W (uniformly bounded under refinement at fixed window geometry).

Proof. Work at a fixed regulator level and suppress Reg from the notation. By definition, $K_t(b, \cdot)$ is the endpoint-disintegration kernel obtained by integrating out bulk variables with density proportional to $\exp\{-S(U)\}$ with respect to the product Haar reference. Projecting to the window W yields a conditional density on $u \in G^W$ of the form

$$p(u \,|\, \mathrm{rest}) \ = \ \frac{\exp\{-S_W(u; \mathrm{rest})\}}{\int_{G^W} \exp\{-S_W(u'; \mathrm{rest})\} \, \mathrm{Haar}_W(\,\mathrm{d}u')}, \tag{208}$$

where rest denotes all non-window variables and $S_W(u; \mathrm{rest})$ is the sum of the finitely many local action terms that depend on the window variables. Finite-range locality implies that only finitely many plaquette/local terms contribute to S_W, and their number N_W depends only on the window geometry (in lattice units), not on the UV refinement.

Because G is compact and the local potentials are continuous, each contributing local term is globally bounded in absolute value by $\beta\|\phi\|_\infty$, hence there exists $M_W(\beta) < \infty$ with $M_W(\beta) \leq N_W \, \beta\|\phi\|_\infty$ such that

$$\sup_{u \in G^W} \sup_{\mathrm{rest}} \big| S_W(u; \mathrm{rest}) \big| \ \leq \ M_W(\beta).$$

From (208) we obtain, for all $u \in G^W$,

$$p(u \,|\, \mathrm{rest}) \ \geq \ \frac{e^{-M_W}}{\int_{G^W} e^{M_W} \, \mathrm{Haar}_W(\,\mathrm{d}u')} \ = \ e^{-2M_W}.$$

Therefore $p(\cdot \,|\, \mathrm{rest}) \geq e^{-2M_W} \, \mathrm{Haar}_W(\cdot)$, and integrating out the remaining variables preserves the lower bound, yielding (206) with $\varepsilon_W = e^{-2M_W}$. □

F.2 From local Doeblin to cylindrical contraction

Fix W and choose the "low-mode" projection Π_m of Section 2 to coincide with Π_W. Since the projected cost $d_\eta(u, v) = 1 \wedge \eta \, d_{\mathrm{proj}}(u, v)$ is bounded by 1, we use the total-variation norm in the convention

$$\|\mu - \nu\|_{\mathrm{TV}} := \sup_{\|f\|_\infty \leq 1} \Big| \int f \, \mathrm{d}(\mu - \nu) \Big|,$$

so that $\|\delta_x - \delta_y\|_{\mathrm{TV}} = 2$ for $x \neq y$. Define, for probability measures μ, ν on $\mathcal{B}_{t,\mathrm{Reg}}$,

$$W_{1,\eta}^{\Pi_m}(\mu, \nu) \ := \ W_{1,\eta}\big((\Pi_m)_\# \mu, \ (\Pi_m)_\# \nu\big),$$

where $W_{1,\eta}$ is the Wasserstein distance on the (finite-dimensional) range of Π_m associated with the bounded cost $d_\eta(u,v) := 1 \wedge \eta\, d_{\mathrm{proj}}(u,v)$, where d_{proj} is any fixed distance on the (finite-dimensional) range of Π_m (in the Wilson track: product geodesic on G^W). Since $d_\eta \le 1$, we have the elementary domination by total variation on the projected space: for any two probability measures μ, ν on $\mathcal{B}_{t,\mathrm{Reg}}$,

$$W_{1,\eta}^{\Pi_m}(\mu,\nu) \;\le\; \big\|(\Pi_m)_\# \mu - (\Pi_m)_\# \nu\big\|_{\mathrm{TV}}. \tag{209}$$

By Lemma F.1, $(\Pi_m)_\# K_{t,\mathrm{Reg}}$ has a Doeblin minorisation with constant ε_W. A standard coupling argument (or the elementary Doeblin contraction lemma) gives, for all $n \ge 0$ and all x, y,

$$\big\|(\Pi_m)_\# \delta_x K^n_{t,\mathrm{Reg}} - (\Pi_m)_\# \delta_y K^n_{t,\mathrm{Reg}}\big\|_{\mathrm{TV}} \;\le\; 2(1-\varepsilon_W)^n.$$

Combining this with (209) yields (10) with $C_K = 2$ and $\lambda_K = 1 - \varepsilon_W$, uniformly in Reg.

G Transfer-side L^2 mixing in a small-β / strong-coupling corridor

This appendix records the Wilson-intrinsic corridor mechanism used in Route B: a transfer-side L^2 contraction for the *Euclidean one-slab transfer kernel* $K_{t,\mathrm{Reg}}$ of Definition 2.7. In the Wilson lattice track the boundary state space is the compact manifold $\mathsf{B}_\partial = G^{\mathcal{E}_\partial}$ equipped with product Haar reference. No Gaussian bridge decomposition, harmonic extension, Dirichlet-to-Neumann operator, or additive linear structure is invoked.

Corridor regime. Fix $t > 0$ and a spatial size L. Assume that the Wilson finite-range regulator family satisfies the locality input Assumption H.2 (verified for Wilson lattice Yang–Mills in Lemma H.4) and that the coupling lies in a high-temperature/KP corridor $0 < \beta \le \beta_\star(t,L)$ in which the polymer/cluster expansion of Appendix H converges uniformly in UV refinement at fixed (t,L).

In this corridor, the cross-slab polymer mechanism of Section H.4 yields a Wilson contraction constant $q_W(t,L) < 1$ and the operator norm bound (225). This operator-theoretic contraction is the only input from the corridor needed to deduce L^2 mixing and a transfer spectral gap for $K_{t,\mathrm{Reg}}$.

G.1 KP corridor output for the Wilson transfer

The next theorem packages the exact Wilson-side output of the KP/cluster-expansion corridor that is used in the main theorem package.

Theorem G.1 (KP corridor: Wilson cross-slab contraction)**.** *Fix $t > 0$ and L. There exists $\beta_\star = \beta_\star(t,L) > 0$ such that for all regulators* Reg *in the Wilson finite-range family and all $0 < \beta \le \beta_\star$:*

(a) (KP corridor and uniform activity majorant.) *The polymer activities and incompatibility graph of Appendix H satisfy the Kotecký–Preiss criterion uniformly in the UV refinement at fixed (t,L), so that all cluster expansions used there converge absolutely with regulator-uniform bounds.*

(b) (Cross-slab maximal correlation bound.) *For the endpoint law $\kappa_{t,\mathrm{Reg}}$ (Definition 2.3) and its marginal $\pi_{t,\mathrm{Reg}}$ (Definition 2.4), Lemma H.8 provides constants $C_{\mathrm{cross}}(t,L) < \infty$ and $m_{\mathrm{cross}}(t,L) > 0$ such that for all $f = f(b^-) \in L^2(\pi_{t,\mathrm{Reg}})$ and $g = g(b^+) \in L^2(\pi_{t,\mathrm{Reg}})$ with $\mathbb{E}_{\pi_{t,\mathrm{Reg}}} f = \mathbb{E}_{\pi_{t,\mathrm{Reg}}} g = 0$,*

$$\big|\mathrm{Cov}_{\kappa_{t,\mathrm{Reg}}}(f(b^-), g(b^+))\big| \;\le\; q_W(t,L)\, \|f\|_{L^2(\pi_{t,\mathrm{Reg}})}\, \|g\|_{L^2(\pi_{t,\mathrm{Reg}})}, \qquad q_W(t,L) \le C_{\mathrm{cross}} e^{-m_{\mathrm{cross}} t} < 1, \tag{210}$$

uniformly in UV refinement at fixed (t,L).

(c) (Transfer operator norm contraction.) *Corollary H.9 yields the mean-zero L^2 operator norm bound*

$$\|K_{t,\mathrm{Reg}}\|_{L_0^2(\pi_{t,\mathrm{Reg}})\to L_0^2(\pi_{t,\mathrm{Reg}})} \;\le\; q_W(t,L) \;<\; 1, \tag{211}$$

uniformly in UV refinement at fixed (t,L).

Proof. Item (a) is the KP verification package in Appendix H (Theorem H.1 together with Lemma H.6 and Lemma H.5). Item (b) is Lemma H.8 and item (c) is Corollary H.9. □

G.2 Uniform L^2 mixing and a transfer gap in the corridor

Corollary G.2 (Uniform transfer L^2 mixing in the corridor)**.** *Fix $t > 0$ and L and assume $0 < \beta \le \beta_\star(t,L)$ lies in the KP corridor of Theorem G.1. Then for every regulator* Reg *and every $f \in L^2(\pi_{t,\mathrm{Reg}})$,*

$$\|K^n_{t,\mathrm{Reg}}f - \pi_{t,\mathrm{Reg}}(f)\|_{L^2(\pi_{t,\mathrm{Reg}})} \;\le\; q_W(t,L)^n\,\|f - \pi_{t,\mathrm{Reg}}(f)\|_{L^2(\pi_{t,\mathrm{Reg}})}, \qquad \forall n \in \mathbb{N}, \tag{212}$$

with the same $q_W(t,L) < 1$ as in (211), uniformly in UV refinement at fixed (t,L). In particular, since $K_{t,\mathrm{Reg}}$ is reversible with respect to $\pi_{t,\mathrm{Reg}}$ (Proposition 11.7), its discrete-time L^2 spectral gap on mean-zero functions satisfies

$$\mathrm{gap}_{L^2}(K_{t,\mathrm{Reg}}) \;\ge\; 1 - q_W(t,L) \;>\; 0. \tag{213}$$

Proof. By Corollary H.9, $K_{t,\mathrm{Reg}}$ is a contraction on $L_0^2(\pi_{t,\mathrm{Reg}})$ with operator norm at most $q_W(t,L)$. Iterating the bound gives (212). Because $K_{t,\mathrm{Reg}}$ is reversible, it acts as a self-adjoint contraction on $L^2(\pi_{t,\mathrm{Reg}})$, and therefore $\sup\{\lambda : \lambda \in \mathrm{Spec}(K_{t,\mathrm{Reg}}|_{L_0^2})\} \le \|K_{t,\mathrm{Reg}}\|_{L_0^2\to L_0^2} \le q_W(t,L)$. This implies (213). □

H Kotecký–Preiss polymer expansion for the corridor bounds

This appendix proves Theorem G.1. We isolate the exact output needed for the main paper: uniform C^2 bounds on the boundary effective interaction and a Wilson-intrinsic cross-slab maximal-correlation/transfer-contraction bound.

H.1 Abstract polymer expansion and the KP criterion

Let $\mathcal{P}$ be a collection of finite "polymers" (finite subsets of a countable set), with an incompatibility relation $\gamma \not\sim \gamma'$ (typically: overlap in space–time). Given activities $w(\gamma) \in \mathbb{C}$, define the formal partition function

$$Z \;:=\; \sum_{\Gamma\ \text{compatible}} \prod_{\gamma\in\Gamma} w(\gamma),$$

where the sum runs over finite compatible families.

Fix a weight $a : \mathcal{P} \to [0,\infty)$. The Kotecký–Preiss condition is

$$\sup_{\gamma\in\mathcal{P}} \frac{1}{a(\gamma)} \sum_{\gamma'\not\sim\gamma} |w(\gamma')|\, e^{a(\gamma')} \;\le\; 1. \tag{214}$$

Theorem H.1 (KP cluster expansion)**.** *Assume* (214)*. Then $Z \neq 0$ and $\log Z$ admits an absolutely convergent cluster expansion*

$$\log Z \;=\; \sum_{\Gamma \text{ connected}} \phi(\Gamma) \prod_{\gamma\in\Gamma} w(\gamma),$$

where the sum runs over finite connected clusters and $\phi(\Gamma)$ are the Ursell coefficients (see e.g. [72, 73]). Moreover, for every $\gamma_0 \in \mathcal{P}$,

$$\sum_{\Gamma\ni\gamma_0} |\phi(\Gamma)| \prod_{\gamma\in\Gamma} |w(\gamma)| \;\le\; a(\gamma_0), \tag{215}$$

and all derivatives of $\log Z$ with respect to parameters entering the activities are obtained by termwise differentiation, with the same type of majorant.

Proof. For a compatible family $\Gamma = \{\gamma_1, \dots, \gamma_n\}$, write $W(\Gamma) := \prod_{i=1}^n w(\gamma_i)$. The Mayer expansion expresses $\log Z$ as (see e.g. [72, 73])

$$\log Z \;=\; \sum_{\Gamma \text{ connected}} \phi(\Gamma)\, W(\Gamma),$$

where $\phi(\Gamma)$ are Ursell coefficients (Möbius inversion on the lattice of partitions). By the tree–graph inequality (see e.g. [70, 72]) (Penrose) one has the bound

$$|\phi(\Gamma)| \;\le\; \sum_{T \text{ tree on } \{1,\dots,n\}} \prod_{(i,j)\in T} \mathbf{1}_{\gamma_i \not\sim \gamma_j},$$

see e.g. [73, 70, 71]. Fix $\gamma_0 \in \mathcal{P}$ and sum over all connected clusters containing γ_0. Root each tree at the vertex corresponding to γ_0 and sum inductively over descendants using the KP condition (214); this yields the majorant (215). Absolute convergence implies $Z \neq 0$. Termwise differentiation with respect to parameters entering the activities is justified by dominated convergence under the same majorant; see [73, 71]. $\square$

H.2 Polymer representation of the boundary effective interaction

We apply Theorem H.1 to the *conditional slab partition functions* entering the disintegration/log–Laplace representation of the boundary law. Recall from (21) that for each boundary datum b one may write

$$Z_{t,\mathrm{Reg}}(b) \;=\; \mathbb{E}\big[\exp\{-\mathcal{V}_{t,\mathrm{Reg}}(b;\Xi)\}\big],$$

where Ξ denotes the interior/bulk degrees of freedom under a convenient reference law $\mathbb{P}$ (depending on Reg), and $\mathcal{V}_{t,\mathrm{Reg}}(b;\Xi)$ is a bulk interaction functional depending on b only through the boundary constraint. The centred one-boundary potential is $U_{t,\mathrm{Reg}}(b) = -\log Z_{t,\mathrm{Reg}}(b) + \log Z_{t,\mathrm{Reg}}(0)$, and the interaction contribution $V_{t,\mathrm{Reg}}$ in the boundary potential $W_{t,\mathrm{Reg}} = W^{(0)}_{t,\mathrm{Reg}} + V_{t,\mathrm{Reg}}$ is obtained from $U_{t,\mathrm{Reg}}$ after separating the quadratic part $W^{(0)}$. In particular, uniform control of the first and second Fréchet derivatives of $b \mapsto \log Z_{t,\mathrm{Reg}}(b)$ implies the desired uniform C^2 bounds on $V_{t,\mathrm{Reg}}$.

Locality input. The KP/cluster expansion is applied to the *local bulk functional* $\mathcal{V}_{t,\mathrm{Reg}}(b;\Xi)$, not to the already-integrated boundary effective potential (which is typically nonlocal in b). We use the following explicit locality/regularity input for the corridor regulator family.

Assumption H.2 (Local slab decomposition and uniform C^2 bounds)**.** In the corridor regime, the bulk interaction functional admits a finite-range decomposition

$$\mathcal{V}_{t,\mathrm{Reg}}(b;\Xi) \;=\; \beta \sum_{x\in\Lambda_L} \Phi_x^{(\mathrm{Reg})}(b;\Xi), \tag{216}$$

where the coupling/temperature parameter β is factored outside the local pieces. Each $\Phi_x^{(\mathrm{Reg})}$ depends only on (b,Ξ) in a fixed neighbourhood of x (independent of the UV cutoff), is C^2 in the boundary variable b on the boundary manifold (in local coordinates; no cutoff is imposed), and satisfies the uniform bounds

$$\sup_{\mathrm{Reg}}\sup_{b,\Xi}\Big(\; |\Phi_x^{(\mathrm{Reg})}(b;\Xi)| + \|D\Phi_x^{(\mathrm{Reg})}(b;\Xi)\| + \|D^2\Phi_x^{(\mathrm{Reg})}(b;\Xi)\|_{\mathrm{op}} \;\Big) \;\le\; C_\Phi(t,L). \tag{217}$$

Moreover, the locality neighbourhood graph on Λ_L induced by the interaction range has uniformly bounded degree under UV refinement at fixed (t,L).

Remark H.3. Assumption H.2 is satisfied for standard finite-range slab regulators globally (without any small-field cutoff): local plaquette terms (and finitely many local counterterms) depend on only finitely many link variables, have bounded derivatives globally on the compact configuration space for compact G, and each degree of freedom participates in a uniformly bounded number of local terms under UV refinement at fixed (t,L).

Lemma H.4 (Verification of Assumption H.2 for Wilson lattice Yang–Mills)**.** *Consider a Wilson lattice Yang–Mills slab regulator at fixed (t,L) with compact gauge group G, Wilson plaquette action, and (optionally) finitely many additional local counterterms of the same finite range. Take $\mathbb{P}$ to be the product Haar law on interior link variables (with boundary links fixed to b) and let Ξ denote the interior links.*

Then Assumption H.2 holds with $C_\Phi(t,L)<\infty$ that is uniform in the UV refinement and auxiliary truncations at fixed (t,L).

Proof. Each Wilson plaquette term (and each allowed local counterterm) depends only on the link variables in a uniformly bounded neighbourhood (finite interaction range, independent of UV refinement). Since G is a compact smooth manifold, fix a finite smooth atlas and write these local terms in local coordinates only as a *notational convenience*: the resulting first and second derivatives are uniformly bounded on each chart, hence globally. Equivalently, the intrinsic first and second derivatives with respect to left/right–invariant vector fields on G are uniformly bounded.

Under UV refinement at fixed (t,L), each boundary link participates in at most a uniformly bounded number of such local terms (bounded coordination number). Combining finite range with the uniform C^2 bounds yields the decomposition and the constants in Assumption H.2, with bounds depending only on (t,L,G) and the chosen local action/counterterm list, and uniform in the UV refinement and auxiliary truncations. □

Polymerisation of $Z_{t,\mathrm{Reg}}(b)$. Define the Mayer functions $F_x(b;\Xi) := \exp(-\beta\Phi_x^{(\mathrm{Reg})}(b;\Xi)) - 1$ and expand

$$\exp\Big(-\beta\sum_x \Phi_x^{(\mathrm{Reg})}(b;\Xi)\Big) \;=\; \prod_x (1+F_x(b;\Xi)).$$

Using the product structure of $\mathbb{P}$ and grouping terms by connected components in the locality/overlap graph yields a polymer model with polymers $\gamma\subset\Lambda_L$ connected in the neighbourhood graph and (boundary-dependent) activities

$$w_b(\gamma) \;:=\; \mathbb{E}\Big[\prod_{x\in\gamma} F_x(b;\Xi)\Big], \tag{218}$$

where the expectation is under $\mathbb{P}$ (with boundary fixed to b). The resulting polymer partition function equals $Z_{t,\mathrm{Reg}}(b)$ (up to the harmless overall normalisation absorbed into the centering at $b=0$), and therefore $V_{t,\mathrm{Reg}}(b) = -\log Z_{t,\mathrm{Reg}}(b) + \log Z_{t,\mathrm{Reg}}(0)$ is controlled by the convergent cluster expansion for $\log Z_{t,\mathrm{Reg}}(b)$.

Lemma H.5 (Counting connected polymers in bounded degree)**.** *Let $(\Lambda, \sim)$ be a graph with maximum degree $\Delta < \infty$. Then the number of connected subsets $\gamma \subset \Lambda$ of size n containing a fixed vertex x is bounded by*

$$N_n(x) \;\le\; (e\Delta)^{n-1}, \qquad n \ge 1.$$

Consequently, for any finite set $A \subset \Lambda$, the number of connected γ of size n that intersect A is at most $|A|\,(e\Delta)^{n-1}$.

Proof. Fix x and let γ be connected of size n containing x. Choose a spanning tree T on γ rooted at x and orient edges away from the root. There are at most n^{n-2} labelled trees on n vertices (Cayley), and each edge of T can be embedded by choosing one of at most Δ neighbours at each step, giving at most Δ^{n-1} embeddings. Using $n^{n-2} \le (en)^{n-2} \le (e\Delta)^{n-2}$ for $\Delta \ge 2$ (and adjusting constants for $\Delta = 1$) yields $N_n(x) \le (e\Delta)^{n-1}$ up to a harmless universal factor absorbed into $e\Delta$. For the second claim, choose $x \in A \cap \gamma$ and sum over $x \in A$. ☐

Lemma H.6 (Activity bounds and KP smallness)**.** *There is a constant $C = C(t, L)$, independent of the UV cutoff and auxiliary truncations, such that for all polymers γ and all b in local coordinates (for notation only; no cutoff is imposed),*

$$|w_b(\gamma)| \;\le\; (C\beta)^{|\gamma|}.$$

Consequently, choosing $a(\gamma) = |\gamma|$ in (214), there exists $\beta_\star = \beta_\star(t, L) > 0$ such that the KP condition holds uniformly in b *for all $0 < \beta \le \beta_\star$.*

Proof. Using $|e^{-u} - 1| \le |u| e^{|u|}$ and (217), we have

$$|F_x(b; \Xi)| \;\le\; C_\Phi\, \beta\, e^{C_\Phi \beta} \qquad \text{uniformly in Reg},\ b,\ \Xi.$$

Hence for every polymer γ,

$$|w_b(\gamma)| \;=\; \Big|\mathbb{E}\Big[\prod_{x\in\gamma} F_x(b;\Xi)\Big]\Big| \;\le\; \mathbb{E}\Big[\prod_{x\in\gamma} |F_x(b;\Xi)|\Big] \;\le\; \prod_{x\in\gamma} \|F_x\|_\infty \;\le\; \big(C_\Phi\, \beta\, e^{C_\Phi\beta}\big)^{|\gamma|}.$$

Absorb $e^{C_\Phi \beta}$ into the constant for $0 < \beta \le 1$ to obtain the claimed bound. The KP smallness condition then follows by combining this bound with the bounded-degree polymer counting estimate of Lemma H.5. ☐

H.3 Derivative bounds and the corridor constants

By Theorem H.1 and Lemma H.6, the polymer expansion for $\log Z$ converges absolutely for $0 < \beta \le \beta_\star$, uniformly in Reg at fixed (t, L). Differentiating termwise in the boundary field yields uniform C^2 bounds.

Lemma H.7 (Uniform C^2 bounds)**.** *For $0 < \beta \le \beta_\star$ there is a constant $C_{\mathrm{Hess}}(t, L)$ such that, in local coordinates (for notation only; no cutoff is imposed),*

$$\sup_{b\in\mathsf{H}} \big\|\Pi_0\, D^2 V_{t,\mathrm{Reg}}(b)\, \Pi_0\big\|_{\mathrm{op}} \;\le\; C_{\mathrm{Hess}}(t, L)\, \beta, \tag{219}$$

uniformly in the UV cutoff and auxiliary truncations at fixed (t, L).

Proof. Differentiate the cluster expansion termwise (Theorem H.1) and use the uniform derivative bounds in (217). Each derivative brings down at most one extra local factor per site, so the same KP majorant applies, producing a convergent series with total size $O(\beta)$ (the first nontrivial term is linear in β). Summing over clusters gives (219). $\square$

H.4 Cross-slab correlation decay and a Wilson-intrinsic contraction constant

This subsection records a Wilson-intrinsic replacement for any linear/harmonic-extension contraction constant: the transfer-side L^2 contraction is bounded by a *maximal correlation* between the two boundary traces, and in the KP corridor this maximal correlation is controlled by polymer clusters that must cross the slab.

Lemma H.8 (Cross-slab correlation decay / maximal correlation bound)**.** *Fix $t > 0$ and L and consider a Wilson finite-range slab regulator at inverse temperature β. Let $\kappa_{t,\mathrm{Reg}}(\mathrm{d}b^-, \mathrm{d}b^+)$ be the endpoint law of Definition 2.3 on $\mathsf{B}_\partial \times \mathsf{B}_\partial$, and let $\pi_{t,\mathrm{Reg}}$ denote its marginal (Definition 2.4). Assume the KP corridor hypotheses of Appendix H (in particular, the activity majorant of Lemma H.6 and the KP convergence of Theorem H.1) hold uniformly in the UV refinement at fixed (t, L), and that the coupling lies in the corridor $0 < \beta \le \beta_\star(t, L)$ provided by Lemma H.6.*

Then there exist constants $C_{\mathrm{cross}} = C_{\mathrm{cross}}(t, L)$ and $m_{\mathrm{cross}} = m_{\mathrm{cross}}(t, L) > 0$, which may depend on (t, L, G) and the corridor choice (e.g. the bound $0 < \beta \le \beta_\star(t, L)$) but are uniform in the regulator Reg *(and hence independent of UV refinement and auxiliary truncations), such that for all $f = f(b^-) \in L^2(\pi_{t,\mathrm{Reg}})$ and $g = g(b^+) \in L^2(\pi_{t,\mathrm{Reg}})$ with $\mathbb{E}_{\pi_{t,\mathrm{Reg}}} f = \mathbb{E}_{\pi_{t,\mathrm{Reg}}} g = 0$,*

$$\left|\mathrm{Cov}_{\kappa_{t,\mathrm{Reg}}}(f(b^-), g(b^+))\right| \;\le\; q_W(t, L)\, \|f\|_{L^2(\pi_{t,\mathrm{Reg}})}\, \|g\|_{L^2(\pi_{t,\mathrm{Reg}})}, \tag{220}$$

where, in this corridor regime $0 < \beta \le \beta_\star(t, L)$, one may take

$$q_W(t, L) \;\le\; C_{\mathrm{cross}}\, e^{-m_{\mathrm{cross}} t} \;<\; 1, \tag{221}$$

uniformly in the UV refinement at fixed (t, L).

Proof. Since the Wilson boundary state space $\mathsf{B}_\partial = G^{E_\partial}$ is compact and all fixed-regulator endpoint laws have strictly positive smooth densities with respect to Haar measure (Corollary 13.14), the endpoint law is absolutely continuous with respect to the product of its marginals.

Write $m(\mathrm{d}b)$ for product Haar on B_∂. As in Appendix H.2, the one-endpoint and two-endpoint conditional partition functions admit the representations

$$\pi_{t,\mathrm{Reg}}(\mathrm{d}b) \;\propto\; Z_{t,\mathrm{Reg}}(b)\, m(\mathrm{d}b), \qquad \kappa_{t,\mathrm{Reg}}(\mathrm{d}b^-, \mathrm{d}b^+) \;\propto\; Z^{(2)}_{t,\mathrm{Reg}}(b^-, b^+)\, m(\mathrm{d}b^-) m(\mathrm{d}b^+),$$

so that the Radon–Nikodym derivative of $\kappa_{t,\mathrm{Reg}}$ with respect to $\pi_{t,\mathrm{Reg}} \otimes \pi_{t,\mathrm{Reg}}$ can be written as

$$\frac{\mathrm{d}\kappa_{t,\mathrm{Reg}}}{\mathrm{d}(\pi_{t,\mathrm{Reg}} \otimes \pi_{t,\mathrm{Reg}})}(b^-, b^+) \;=\; \frac{r_{t,\mathrm{Reg}}(b^-, b^+)}{\Theta_{t,\mathrm{Reg}}}, \qquad r_{t,\mathrm{Reg}}(b^-, b^+) \;\coloneqq\; \frac{Z^{(2)}_{t,\mathrm{Reg}}(b^-, b^+)}{Z_{t,\mathrm{Reg}}(b^-)\, Z_{t,\mathrm{Reg}}(b^+)}, \tag{222}$$

where $\Theta_{t,\mathrm{Reg}} \coloneqq \mathbb{E}_{\pi_{t,\mathrm{Reg}} \otimes \pi_{t,\mathrm{Reg}}}[r_{t,\mathrm{Reg}}]$ is the normalising constant.

Step 1: KP/cluster expansion isolates the cross-slab part of $\log r_{t,\mathrm{Reg}}$. Applying the same KP polymer representation as in Appendix H.2 to $Z^{(2)}_{t,\mathrm{Reg}}(b^-, b^+)$ (treating (b^-, b^+) as boundary parameters) and subtracting the one-endpoint expansions for $\log Z_{t,\mathrm{Reg}}(b^-)$ and $\log Z_{t,\mathrm{Reg}}(b^+)$ yields a convergent cluster expansion of the form

$$\log r_{t,\mathrm{Reg}}(b^-, b^+) \;=\; \sum_{\substack{\Gamma \text{ connected cluster} \\ \Gamma \cap \partial_- \neq \emptyset,\ \Gamma \cap \partial_+ \neq \emptyset}} \phi(\Gamma) \prod_{\gamma \in \Gamma} w_{b^-, b^+}(\gamma), \tag{223}$$

where $w_{b^-,b^+}(\gamma)$ are the polymer activities in the two-endpoint model and ∂_- (resp. ∂_+) denotes the boundary layer (sites within the fixed interaction range of the bottom, resp. top, time boundary). Only clusters intersecting *both* boundary layers survive in the difference, hence the "cross-slab" restriction in (223).

Step 2: cross-slab clusters must be long, hence are exponentially suppressed. Because the interaction range is finite (Assumption H.2) and the slab has thickness t, any connected cluster intersecting both boundary layers must contain an incompatibility chain of length at least $c_0(t,L)\,t$ (measured in the local-neighbourhood graph), for some $c_0(t,L) > 0$ depending only on local slab geometry (and in particular *not* on UV refinement at fixed (t,L)). Using the activity majorant from Lemma H.6 and polymer counting (Lemma H.5) inside the KP bound (215) gives an exponential tail for the absolute cluster sum: there exist $C_{\text{cross}}, m_{\text{cross}} > 0$ (depending only on (t,L) and corridor constants) such that

$$\sup_{b^-,b^+} \sum_{\Gamma \text{ cross slab}} |\phi(\Gamma)| \prod_{\gamma\in\Gamma} |w_{b^-,b^+}(\gamma)| \;\le\; C_{\text{cross}}\, e^{-m_{\text{cross}} t}. \tag{224}$$

The supremum over boundary traces is harmless: by compactness of $\mathsf{B}_\partial = G^{E_\partial}$ and finite-range locality, the local factors entering $w_{b^-,b^+}(\gamma)$ are uniformly bounded for all fixed (b^-, b^+). Moreover, the KP activity majorant in Lemma H.6 and the counting bound in Lemma H.5 are uniform in the regulator Reg throughout the corridor $0 < \beta \le \beta_\star(t,L)$, so (224) holds uniformly in boundary conditions and UV refinement at fixed (t,L).

Step 3: pointwise control of the RN derivative. Let $S := \sup_{b^-,b^+} |\log r_{t,\text{Reg}}(b^-,b^+)|$. By (223)–(224), we have $S \le C_{\text{cross}} e^{-m_{\text{cross}} t}$. Hence $r_{t,\text{Reg}} \in [e^{-S}, e^S]$ and also $\Theta_{t,\text{Reg}} \in [e^{-S}, e^S]$. Therefore the density in (222) satisfies

$$\sup_{b^-,b^+} \left| \frac{\mathrm{d}\kappa_{t,\text{Reg}}}{\mathrm{d}(\pi_{t,\text{Reg}} \otimes \pi_{t,\text{Reg}})}(b^-,b^+) \;-\; 1 \right| \;\le\; e^{2S} - 1 \;\le\; (e^{2S})\,(2S) \;\le\; 2e^{2S}\, C_{\text{cross}} e^{-m_{\text{cross}} t}.$$

Absorb the harmless factor $2e^{2S}$ into the constant (uniformly in UV refinement at fixed (t,L)) to obtain (221).

Step 4: covariance bound. Let $R(b^-,b^+) := \frac{\mathrm{d}\kappa_{t,\text{Reg}}}{\mathrm{d}(\pi_{t,\text{Reg}}\otimes\pi_{t,\text{Reg}})}(b^-,b^+) - 1$. If $\mathbb{E}_{\pi_{t,\text{Reg}}} f = \mathbb{E}_{\pi_{t,\text{Reg}}} g = 0$, then

$$\operatorname{Cov}_{\kappa_{t,\text{Reg}}}(f(b^-), g(b^+)) = \mathbb{E}_{\pi_{t,\text{Reg}}\otimes\pi_{t,\text{Reg}}}\big[f(b^-)g(b^+)\,R(b^-,b^+)\big].$$

By Cauchy–Schwarz and the pointwise bound $\|R\|_\infty \le q_W(t,L)$,

$$\big|\operatorname{Cov}_{\kappa_{t,\text{Reg}}}(f(b^-), g(b^+))\big| \le \|R\|_\infty\, \|f\|_{L^2(\pi_{t,\text{Reg}})}\, \|g\|_{L^2(\pi_{t,\text{Reg}})}.$$

This is (220).

□

Corollary H.9 (Operator norm bound for the Euclidean transfer kernel)**.** *Under the assumptions of Lemma H.8, let $K_{t,\text{Reg}}$ be the Euclidean transfer kernel (Definition 2.7) for the endpoint law $\kappa_{t,\text{Reg}}$. Interpreting $K_{t,\text{Reg}}$ as conditional expectation,*

$$\|K_{t,\text{Reg}}\|_{L^2_0(\pi_{t,\text{Reg}})\to L^2_0(\pi_{t,\text{Reg}})} \;\le\; q_W(t,L) \;<\; 1, \tag{225}$$

uniformly in UV refinement at fixed (t,L).

Proof. For $f, g \in L^2_0(\pi_{t,\mathrm{Reg}})$, using the disintegration identity $\kappa_{t,\mathrm{Reg}}(\,\mathrm{d}b^-,\,\mathrm{d}b^+) = \pi_{t,\mathrm{Reg}}(\,\mathrm{d}b^-)\,K_{t,\mathrm{Reg}}(b^-,\,\mathrm{d}b^+)$,

$$\langle K_{t,\mathrm{Reg}} f,\, g\rangle_{L^2(\pi_{t,\mathrm{Reg}})} = \mathbb{E}_{\kappa_{t,\mathrm{Reg}}}[f(b^-)g(b^+)] = \mathrm{Cov}_{\kappa_{t,\mathrm{Reg}}}(f(b^-), g(b^+)).$$

Since cylinder functions generate the boundary σ-algebra, they are dense in $L^2(\pi_{t,\mathrm{Reg}})$; the covariance bound of Lemma H.8 therefore extends from cylinders to all L^2 functions by L^2-approximation and continuity. Apply Lemma H.8 and take the supremum over g with $\|g\|_{L^2(\pi_{t,\mathrm{Reg}})} = 1$ to obtain $\|K_{t,\mathrm{Reg}} f\|_{L^2(\pi_{t,\mathrm{Reg}})} \le q_W(t, L)\,\|f\|_{L^2(\pi_{t,\mathrm{Reg}})}$. This proves (225). □

I Thermodynamic limit $L \to \infty$ in the KP corridor

This section supplies a genuine spatial thermodynamic limit within the strong-coupling/KP corridor. Throughout we fix $t > 0$, a compact connected Lie group G, and a Wilson finite-range regulator family on slabs. The spatial volume parameter L varies while the interaction range (in lattice units) is fixed.

I.1 Local topology and cylinder observables

Let $\mathsf{H}_{\partial,L} := G^{E_\partial(t,L,\mathrm{Reg})}$ be the boundary link space at spatial size L (with Haar reference) and let $\mathsf{X}_{t,L} := G^{E(t,L,\mathrm{Reg})}$ be the full slab link space. A *local/cylinder observable* is a bounded measurable function depending only on a finite set of links (equivalently, supported in a fixed finite window $W \subset E$). We say that probability measures $\{\mu_L\}_{L\ge 1}$ on $\mathsf{X}_{t,L}$ converge *locally* if $\mu_L(F)$ converges for every cylinder observable F with support contained in $W \subset E(t, L_0, \mathrm{Reg})$ for some fixed L_0.

I.2 An L-uniform KP corridor

The polymer expansion in Appendix H is formulated in terms of finite-range local pieces and an incompatibility graph. Crucially, both the *local activity bounds* and the *maximal incompatibility degree* depend only on the local geometry of the slab and are therefore independent of the spatial volume L.

Theorem I.1 (Thermodynamic limit in an L-uniform KP corridor)**.** *There exists $\beta_\star^\infty(t) > 0$ such that the following holds. Assume $0 < \beta \le \beta_\star^\infty(t)$ and consider the Wilson slab Gibbs measures $\mu^{(\infty)}_{t,L,\mathrm{Reg}}$ (with periodic spatial boundary conditions and free boundary in Euclidean time) and their boundary marginals $\pi_{t,L,\mathrm{Reg}}$. Then:*

(i) Existence and uniqueness of an infinite-volume slab state. *There exists a unique infinite-volume DLR state $\mu^{(\infty)}_{t,\infty,\mathrm{Reg}}$ on the spatially infinite slab such that for every cylinder observable F there is a limit*

$$\lim_{L\to\infty} \mu^{(\infty)}_{t,L,\mathrm{Reg}}(F) \;=:\; \mu^{(\infty)}_{t,\infty,\mathrm{Reg}}(F).$$

Moreover, the convergence is exponentially fast in the distance of $\mathrm{supp}(F)$ *from the spatial boundary of the finite torus.*

(ii) Limit of boundary laws on finite windows. *For every fixed finite boundary window $W \subset E_\partial$, the projected boundary laws $(\Pi_W)_\# \pi_{t,L,\mathrm{Reg}}$ converge as $L \to \infty$. The limits are consistent under restriction and therefore define a boundary law $\pi_{t,\infty,\mathrm{Reg}}$ on the infinite boundary configuration space by Kolmogorov extension.*

(iii) Limit transfer specification. *Let $\nu_{t,L,\mathrm{Reg}}$ be the endpoint joint law on $(\mathsf{H}_{\partial,L})^2$ and let $K_{t,L,\mathrm{Reg}}$ be the corresponding endpoint-disintegration transfer kernel. For every fixed boundary window W, the disintegrations of the projected endpoint laws converge as $L \to \infty$, yielding a limiting finite-window transfer specification $K^W_{t,\infty,\mathrm{Reg}}$. These finite-window kernels are consistent in W and define a transfer specification $K_{t,\infty,\mathrm{Reg}}$ on the infinite boundary space.*

(iv) Persistence of Euclidean-time clustering for local observables. *For any fixed $\delta \in (0, t/2)$ and any gauge-invariant cylinder observables $\mathcal{O}, \mathcal{P} \in \mathfrak{Obs}^{\mathrm{lat}}_{t,\delta}$ (Definition 2.1) with support contained in a fixed spatial window independent of L, the Euclidean-time clustering bound (152) holds in the $L \to \infty$ limit with the same rate parameter m_* as in finite volume.*

Proof. We prove (i)–(iv) using the KP machinery already developed in Appendix H.

Step 1: L-uniform convergence of the polymer/cluster expansion. Appendix H produces activities $\mathsf{w}(\gamma)$ for polymers γ (finite connected plaquette sets) and a KP majorant of the form

$$|\mathsf{w}(\gamma)| \;\le\; (C_{\mathrm{act}}\beta)^{|\gamma|},$$

together with an incompatibility graph of uniformly bounded degree Δ (both constants depending only on local slab geometry). Therefore the Kotecký–Preiss criterion holds whenever $\beta \le \beta^\infty_\star(t)$ with $\beta^\infty_\star(t) := (e\, C_{\mathrm{act}}\Delta)^{-1}$, and the standard cluster expansion gives absolutely convergent series for $\log Z_{t,L,\mathrm{Reg}}$ and for cumulants of cylinder observables, with bounds independent of L (see Theorem G.1 and the estimates (215)).

Step 2: local limits and DLR uniqueness. Fix a cylinder observable F supported in a window W and write $r_L := \mathrm{dist}(W, \partial\Lambda_L)$ for the spatial distance from W to the boundary scale of the spatial torus (equivalently, to the wrap-around identification). Using the standard cluster-expansion representation for expectations with an insertion (obtained by expanding numerator and denominator and regrouping into connected clusters), one may write

$$\mu^{(\infty)}_{t,L,\mathrm{Reg}}(F) \;=\; \sum_{\Gamma:\, \Gamma\cap W\neq\emptyset} \Phi_F(\Gamma) \prod_{\gamma\in\Gamma} w(\gamma),$$

where the sum runs over finite connected clusters of polymers and the coefficients $\Phi_F(\Gamma)$ are bounded in absolute value by a combinatorial factor depending only on the incompatibility graph. Because the interaction range is finite, any cluster contributing to the difference between two volumes must connect W to the complement of the smaller volume; in particular such a cluster must contain at least $c_0\, r_L$ polymers along an incompatibility chain. The KP bound (215) and the activity majorant from Theorem G.1 therefore give an exponential tail: there exist $C, c > 0$ (independent of L) such that

$$\big|\mu^{(\infty)}_{t,L,\mathrm{Reg}}(F) - \mu^{(\infty)}_{t,L',\mathrm{Reg}}(F)\big| \;\le\; 2\|F\|_\infty\, C\, e^{-c\,\min(r_L, r_{L'})}.$$

Hence $\{\mu^{(\infty)}_{t,L,\mathrm{Reg}}(F)\}_L$ is Cauchy and converges as $L \to \infty$, proving (i). The same local-uniqueness estimate implies uniqueness of the infinite-volume DLR state in the corridor: any two DLR states agree on all cylinder observables and therefore coincide. **Step 3: boundary marginals and transfer specification.** Items (ii) and (iii) follow by applying the same local-Cauchy argument to cylinder observables depending only on boundary links (and to cylinder observables on two time boundaries for endpoint laws), then using standard disintegration on standard Borel spaces. Consistency under restriction in W is inherited from the finite-volume measures, so Kolmogorov extension applies.

Step 4: persistence of Euclidean-time clustering. The clustering bound (152) for local observables is proved in finite volume from transfer-side mixing/decay estimates together with uniform response bounds in the corridor. All ingredients are local in the spatial direction, with constants independent of L in the corridor, so the bound passes to the $L \to \infty$ limit along the convergence of cylinder expectations established above, proving (iv). $\square$

Data availability statement

No datasets were generated or analyzed during the current study.

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