Weak Harris Mixing and Exponential Clustering for Interacting Boundary Laws on Abstract Wiener Spaces

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Abstract

We develop a weak Harris framework for interacting probability measures on abstract Wiener spaces and apply it to exponential clustering (OS4) in Euclidean quantum field theory. Starting from a Gaussian reference measure tilted by a potential on an abstract Wiener space, we formulate quantitative structural hypotheses that encode ellipticity, local C_H^2-regularity, one-sided growth, the existence of a coercive Lyapunov function, and a projected Doeblin minorisation for a discrete-time skeleton of the associated Langevin dynamics. Under these assumptions we prove a weak Harris theorem in an adapted Kantorovich distance $W_1^{(m)}$, combining control of finitely many low modes with a Lyapunov component, and obtain exponential convergence to equilibrium with constants that are uniform in families of models sharing the same structural data. We then introduce an abstract boundary-law and slab-Markov representation for time-separated observables and show that Harris mixing for the boundary chain implies an OS4 exponential clustering bound for Schwinger function covariances, with a strictly positive decay rate $\rho(t) > 0$ that is stable under regulator limits. The results are formulated in a model-independent way so that they can be applied in constructive Euclidean field theories admitting a slab decomposition and an appropriate boundary representation.

1 Introduction

Many constructive Euclidean field theories produce interacting measures on infinite-dimensional configuration spaces by tilting a Gaussian reference law with a non-quadratic potential. A natural way to analyse such measures is to study the Langevin dynamics that preserves them and to use Harris-type ergodic theorems to obtain quantitative mixing. In a Euclidean quantum field theory context, an additional structural feature is the existence of a Markovian "slab" decomposition in Euclidean time, which allows one to represent time-separated observables in terms of boundary data and to translate mixing of the boundary dynamics into exponential clustering (OS4) for Schwinger functions via the Osterwalder–Schrader reconstruction.

The aim of this paper is to isolate a set of analytic hypotheses under which this programme can be carried out in a parameter-uniform way. We work with an abstract Wiener space (B, H, μ^0) and an interacting boundary law

$$\mu(db) \;=\; Z^{-1} \exp\bigl(-U(b)\bigr)\, \mu^0(db),$$

together with the associated boundary Langevin dynamics, and we formulate structural assumptions (A1)–(A4) on the Gaussian covariance and the potential. These assumptions encode uniform ellipticity, local C_H^2-regularity and Lipschitz bounds, a one-sided growth condition, a coercive Lyapunov functional, and the existence of a projected minorisation condition for a discrete-time skeleton of the dynamics. Under these hypotheses we prove a weak Harris theorem in an adapted Kantorovich distance $W_1^{(m)}$ tailored to the Lyapunov function and to a finite-dimensional projection. The existence of the adapted distance and the contraction estimate rests on the general weak Harris theory of Hairer–Mattingly and Eberle [9, 10, 6], while we keep careful track of the dependence of the constants on the structural data.

A key feature of the framework is that all constants in the Lyapunov drift, the small-set minorisation, and the contraction estimates can be chosen *uniformly* for families of models that share the same structural parameters. From the point of view of constructive field theory, this parameter-uniformity is essential: Euclidean measures are typically constructed at finite ultraviolet, infrared, and volume regulators and then passed to a continuum and infinite-volume limit. In order to transfer OS4 from the regulated models to the limiting theory, one needs uniform lower bounds on the decay rate $\rho(t)$ and uniform control on the Harris constants across the regulator family. The abstract assumptions are therefore formulated with such families in mind, and the structural parameters in (A1)–(A4) and (H1)–(H5) are precisely those that remain stable under the limits considered in the companion work [4].

On the Euclidean field theory side, we assume that the interacting measure admits a slab Markov decomposition and that slab-supported observables can be represented as functionals of the boundary data at discrete times, with Lipschitz and growth control in the same metrics that enter the Harris analysis. We show that, under these assumptions, the exponential contraction in $W_1^{(m)}$ implies an OS4-type exponential clustering bound for Schwinger function covariances, with a decay rate $\rho(t) > 0$ that is uniform in any family of models sharing the same structural constants. We stress that this provides a strictly positive decay rate for each fixed slab thickness $t > 0$. Identifying the physical mass gap in the sense of the full Hamiltonian spectrum involves additional optimisation in t and is not addressed in this paper.

Novelty and scope. At a conceptual level, the paper packages two sets of ideas which appear in various guises in the literature into a single reusable framework:

- a weak Harris theorem in an adapted Kantorovich distance on an abstract Wiener space, in the spirit of Hairer–Mattingly and Eberle [9, 6], and formulated so that all constants are explicitly expressed in terms of Lyapunov and minorisation data; and
- an abstract boundary-law language for Euclidean slab decompositions which turns Harris mixing for a boundary chain, plus a Lipschitz boundary representation of slab-supported observables, into the OS4 clustering axiom with a strictly positive rate $\rho(t)$ that is stable in families of models.

The weak Harris part is thus a parameter-uniform variant of known results, adapted to a "mixed" distance that singles out a finite set of low modes and incorporates a Lyapunov functional in a way convenient for constructive field theory. The OS side combines the classical OS reconstruction and Glimm–Jaffe framework [12, 13, 7] with an explicit boundary representation assumption which is verified by standard polymer and quasi-locality methods in concrete models (such as massive scalar fields and Yang–Mills theories) but is treated here at an abstract level.

From the point of view of applications, the framework is designed to serve as a modular "Harris/OS4 engine" for constructive Euclidean models with slab decompositions. In particular, in a companion work [4] it is used to obtain a regulator-uniform OS4 constant for slab boundary laws in four-dimensional SU(N) Yang–Mills theory, while similar hypotheses can also be verified for massive scalar ϕ^4 theory in finite volume using classical constructive estimates and cluster expansion techniques in the spirit of Feldman, Osterwalder and Hurd. In the present paper we keep the

Yang–Mills and scalar verifications at the level of schematic examples and references; the detailed estimates are carried out in model-specific works.

The structure of the paper is as follows. Section 2 sets up the abstract Wiener space framework and states the structural assumptions (A1)–(A4). Section 3 recalls the weak Harris theory of Hairer–Mattingly/Eberle in a form adapted to our setting and explains how it yields the mixing theorem in $W_1^{(m)}$. Section 4 introduces a convenient set of structural hypotheses (H1)–(H5) for interacting boundary laws, shows that they imply (A4), and discusses how they arise in slab boundary laws for Yang–Mills theory and scalar models. Section 5 formulates the boundary representation Assumption 5.1, derives OS4 via slab concatenation and OS reconstruction, and discusses the extension from discrete to continuous times and the relation to the Hamiltonian spectral gap.

2 Analytic Setting and Structural Assumptions

We begin by fixing the analytic framework and recording the structural assumptions on the interacting potential U which enter the Harris mixing theorem. These assumptions are phrased in terms of the geometry of an abstract Wiener space and are independent of any specific model; concrete examples, including slab boundary laws for Yang–Mills theory and scalar fields, will be discussed later.

2.1 Abstract Wiener space and Gaussian reference measure

Let (B, H, μ^0) be an abstract Wiener space in the sense of Gross [8]:

- H is a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$;
- B is a separable real Banach space containing H densely and continuously via an injection $i\colon H \hookrightarrow B$;
- μ^0 is a centred, non-degenerate Gaussian probability measure on $(B, \mathcal{B}(B))$ with Cameron–Martin space H.

We regard H as a densely and continuously embedded subspace of B via the inclusion i, and, with a slight abuse of notation, we write $\|b\|_H$ for $b \in H$ and for its image in B whenever no confusion can arise.

We denote by $C\colon H \to H$ the covariance operator of the Gaussian measure μ^0. In concrete cutoff models (e.g. finite-volume, finite-ultraviolet approximants in constructive QFT) one may think of H as finite-dimensional, with C a symmetric, positive-definite covariance matrix with eigenvalues $\lambda_k > 0$ bounded above and below uniformly in the regulators. The structural role of C in what follows is encoded entirely in the ellipticity bounds below; no genuine infinite-dimensional spectral theory will be used.

We assume a uniform ellipticity bound.

(A1) Gaussian ellipticity. The covariance C of μ^0 satisfies

$$\langle b, C^{-1} b \rangle_H \;\geq\; \kappa_0 \, \|b\|_H^2, \qquad \|C^{-1}\|_{\mathrm{op}} \;\leq\; \kappa_1, \tag{2.1}$$

for some positive constants κ_0, κ_1 which may depend on a fixed slab thickness $t > 0$ but are otherwise fixed (and in applications, uniform in the regulators).

2.2 Interacting potential and boundary law

Let $U\colon B \to \mathbb{R}$ be a measurable function, referred to as the *interacting potential*. We assume U is finite μ^0-almost surely and satisfies $e^{-U} \in L^1(B, \mu^0)$. The interacting boundary law is the probability measure μ on B defined by

$$\mu(db) \;=\; Z^{-1} \exp\big(-U(b)\big)\, \mu^0(db), \qquad Z \;=\; \int_B e^{-U(b)}\, \mu^0(db) \in (0, \infty). \tag{2.2}$$

We denote by $\nabla_H U$ and $D_H^2 U$ the H-gradient and H-Hessian of U, whenever they exist, and view them as elements of H and $\mathcal{L}(H)$, respectively.

2.3 Langevin dynamics and Markov semigroup

Given the data (B, H, μ^0, U), we consider the stochastic differential equation

$$dB_s \;=\; -\nabla_H \Phi(B_s)\, ds + dW_s, \qquad \Phi(b) \;=\; \frac{1}{2}\langle b, C^{-1} b\rangle_H + U(b), \tag{2.3}$$

where C is the covariance operator of μ^0 and W_s is an H-cylindrical Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$. We assume that (2.3) is well-posed in the sense that for each initial datum $B_0 = b \in B$ there exists a unique strong solution $(B_s)_{s\geq 0}$ with continuous paths in B.

The process $(B_s)_{s\geq 0}$ is Markov with respect to its natural filtration, and we write P_s for its transition semigroup:

$$P_s f(b) \;=\; \mathbb{E}\big[f(B_s)\,\big|\, B_0 = b\big], \qquad f \in \mathcal{B}_b(B),$$

where $\mathcal{B}_b(B)$ denotes the bounded Borel measurable functions on B.

Remark 2.1 (On well-posedness and invariance)**.** The well-posedness of (2.3) and the existence of an invariant probability measure of the form (2.2) can be established under various combinations of monotonicity, coercivity and local Lipschitz assumptions on $\nabla_H U$; see, for example, the monographs of Da Prato and Zabczyk [5] and Bogachev [2] and references therein for infinite-dimensional Langevin dynamics. A typical sufficient condition, compatible with our Lyapunov framework, is that the drift $-\nabla_H \Phi$ is locally Lipschitz on H, satisfies a one-sided Lipschitz or monotonicity condition of the form

$$\langle b - \tilde{b}, \nabla_H \Phi(b) - \nabla_H \Phi(\tilde{b})\rangle_H \leq L\|b - \tilde{b}\|_H^2,$$

together with a coercive Lyapunov bound $\mathcal{L}V \leq -\lambda V + C$ for some $V \geq 1$. Under such hypotheses one obtains global existence, uniqueness and invariance of μ. In the abstract development below we do not attempt to optimise such conditions: we simply treat well-posedness of (2.3) and invariance of μ as part of the structural input, and refer to model-specific works (such as [4]) for concrete verifications in particular examples.

2.4 Structural assumptions on the potential

We now state the structural assumptions on U which will be used to prove Harris mixing and OS4. Fix a slab thickness $t > 0$ and a radius $R > 0$. Let

$$B_R \;:=\; \{b \in B : \|b\|_H \leq R\}$$

denote the H-ball of radius R in B.

(A2) Local C^2_H-regularity and Lipschitz bounds. For each $R > 0$, the restriction $U|_{B_R}$ is twice Fréchet differentiable along H with

$$\sup_{b\in B_R} \|D^2_H U(b)\|_{\mathrm{op}} \le C_2(t,R), \qquad \|\nabla_H U(b) - \nabla_H U(\tilde b)\|_H \le C_2(t,R)\,\|b-\tilde b\|_H \tag{2.4}$$

for all $b, \tilde b \in B_R$, where $C_2(t,R)$ is finite and depends only on (t,R).

(A3) One-sided growth bound. There exist nonnegative constants $K_1(t)$ and $K_0(t)$, depending only on t, such that

$$\big|\langle b, \nabla_H U(b)\rangle_H\big| \le K_1(t)\,\|b\|_H + K_0(t), \qquad b \in H. \tag{2.5}$$

Remark 2.2. Assumption (A3) will not be needed in the abstract Harris theorem of Section 3, which uses only the Lyapunov and minorisation structure in (A4) below. It is included here because it arises naturally in model-specific verifications (for example, in the companion Yang–Mills work [4]) and is convenient for controlling the generator acting on Lyapunov functionals.

(A4) Lyapunov drift, projected minorisation, and coercivity. There exist a measurable function $V\colon B \to [1,\infty)$, constants $a \in (0,1)$, $c \ge 0$, and a radius $R > 0$, together with a time step $s_t > 0$, such that the following hold for the discrete-time chain $X_n := B_{ns_t}$ with transition kernel $P := P_{s_t}$:

(i) (*Lyapunov drift*) For all $b \in B$,

$$PV(b) \le a\,V(b) + c. \tag{2.6}$$

(ii) (*Projected Doeblin minorisation on C_R*) There exists a bounded linear map $\Pi_m\colon B \to H$ with finite-dimensional range such that its restriction to H is an orthogonal projection, a probability measure ν on $\Pi_m H$, and a constant $\delta_0 \in (0,1]$ such that for all $b \in C_R := \{V \le R\}$ and all Borel sets $A \subset \Pi_m H$,

$$\mathbb{P}\big(\Pi_m X_1 \in A \,\big|\, X_0 = b\big) \ge \delta_0\,\nu(A). \tag{2.7}$$

(iii) (*Coercive Lyapunov functional*) There exist constants $\alpha_1, \alpha_2 > 0$ and $p \ge 1$ such that

$$\alpha_1(1 + \|b\|_H^2) \le V(b) \le \alpha_2(1 + \|b\|_H^p), \qquad b \in B. \tag{2.8}$$

In the next section we explain how (A1)–(A4) fit into the general weak Harris framework and state the corresponding mixing result in an adapted Kantorovich distance. Section 4 will introduce a concrete set of hypotheses (H1)–(H5) under which (A1)–(A4) can be verified in practice.

2.5 Main abstract Harris result

We now state the main Harris mixing theorem for the discrete-time chain $(X_n)_{n\ge 0}$ defined above. The associated Kantorovich functional will be denoted $W_1^{(m)}$ and constructed in Section 3 by combining the Lyapunov drift and the projected minorisation via the weak Harris theory of Hairer–Mattingly and Eberle.

Theorem 2.3 (Harris mixing in an adapted Kantorovich distance)**.** *Assume* (A1)–(A4) *and let $P := P_{s_t}$ be the skeleton kernel of the boundary Langevin dynamics. Fix an integer $m \ge 1$ and the projection Π_m appearing in* (A4)(ii)*. Then there exist:*

- *a bounded distance-like function* $d\colon B \times B \to [0,\infty)$*, and*
- *constants* $\kappa \in (0,1)$ *and* $C < \infty$*,*

depending only on the structural data in (A1)–(A4) *and on* m*, such that if* $W_1^{(m)}$ *denotes the Kantorovich distance associated with d (see Definition 3.5 below), the following hold:*

(i) For all probability measures ν_1, ν_2 *on* B *with* $\nu_i(V) < \infty$*,*

$$W_1^{(m)}(\nu_1 P, \nu_2 P) \;\le\; \kappa\, W_1^{(m)}(\nu_1, \nu_2).$$

(ii) The Markov chain $(X_n)_{n\ge 0}$ *admits a unique invariant probability measure* μ *on* B *with* $\mu(V) < \infty$*.*

(iii) There exists a constant $C < \infty$ *such that for all probability measures* ν *on* B *with* $\nu(V) < \infty$ *and all* $n \ge 0$*,*

$$W_1^{(m)}(\nu P^n, \mu) \;\le\; C\,\kappa^n\,(1 + \nu(V)).$$

Moreover, the constants C *and* κ *and the function* d *can be chosen uniformly over families of potentials* U *which share the same Lyapunov, minorisation and coercivity parameters in* (A4) *and the same constants in* (A1)–(A3)*.*

Remark 2.4. By a *distance-like* function we mean a symmetric function $d\colon B \times B \to [0,\infty)$ which vanishes on the diagonal and satisfies the triangle inequality, but which need not separate points. The existence of the adapted distance-like function d and the contraction property in item (i) are consequences of the weak Harris theory developed in Hairer–Mattingly [9, 10] and Eberle [6], applied with the Lyapunov function V and the projected minorisation in (A4). Section 3 explains this reduction. Items (ii) and (iii) are standard ergodic consequences of the contraction in $W_1^{(m)}$ combined with the Lyapunov control of moments; uniqueness of the invariant law is provided by the general weak Harris theorem itself.

3 An Abstract Weak Harris Theorem in $W_1^{(m)}$

In this section we explain how Assumptions (A1)–(A4) fit into the general weak Harris framework of Hairer–Mattingly and Eberle and indicate how this yields Theorem 2.3. For the abstract argument we no longer need the explicit Wiener structure; we only use the Markov kernel P on a Polish space B, a Lyapunov function V and a finite-dimensional projection Π_m as in (A4).

3.1 Lyapunov iteration and moment bounds

We first record the standard consequences of the Lyapunov drift condition.

Lemma 3.1 (Iterated Lyapunov bound)**.** *Suppose* (A4)(i) *holds. Then for all* $n \in \mathbb{N}$ *and all* $x \in B$*,*

$$P^n V(x) \;\le\; a^n V(x) + \frac{c}{1-a}. \tag{3.1}$$

Consequently, for any initial law $\mu_0 \in \mathcal{P}(B)$ *with* $\int V\, d\mu_0 < \infty$*,*

$$\int V\, d(\mu_0 P^n) \;\le\; a^n \int V\, d\mu_0 + \frac{c}{1-a}, \qquad n \in \mathbb{N}. \tag{3.2}$$

Proof. Iterating (2.6) gives

$$P^{n+1}V(x) = P(P^nV)(x) \le a\,P^nV(x) + c$$

for all $n \ge 0$. Solving this scalar recursion yields

$$P^nV(x) \le a^nV(x) + c\sum_{k=0}^{n-1} a^k \le a^nV(x) + \frac{c}{1-a}.$$

This proves (3.1). Integrating (3.1) against μ_0 gives (3.2). □

The following corollary controls the expected growth of the Lyapunov part under one step of the chain.

Corollary 3.2 (One-step control of V under P)**.** *Under* (A4)(i)*, for any $x, y \in B$ we have*

$$\mathbb{E}\big[\, V(X_1) + V(Y_1) \,\big|\, X_0 = x,\, Y_0 = y\big] \;\le\; a\big(V(x) + V(y)\big) + 2c, \tag{3.3}$$

where (X_n) and (Y_n) are copies of the Markov chain with transition kernel P.

Proof. By linearity and (2.6),

$$\mathbb{E}\big[V(X_1) + V(Y_1) \mid X_0 = x, Y_0 = y\big] = PV(x) + PV(y) \le a\big(V(x) + V(y)\big) + 2c.$$

□

3.2 Adapted distance and weak Harris theorem

We now recall, in a form suitable for our purposes, the weak Harris theorem of Hairer–Mattingly and Eberle and explain how it produces the adapted distance-like function d and the contraction in $W_1^{(m)}$ appearing in Theorem 2.3.

Let Π_m and $C_R := \{V \le R\}$ be as in (A4)(ii), and define a truncated projected semimetric

$$\delta(x, y) := \big\|\Pi_m x - \Pi_m y\big\|_H \wedge 1, \qquad x, y \in B. \tag{3.4}$$

Assumption (A4)(ii) is precisely a Doeblin minorisation for the projected chain $\Pi_m X_1$ on the set C_R with respect to the reference measure ν and constant $\delta_0 \in (0, 1]$.

The weak Harris theory (see in particular [9, 10, 6]) shows that a Lyapunov drift condition of the form (2.6), together with such a minorisation for a bounded "base" semimetric δ, yields an exponentially contracting Kantorovich distance built from a suitable modification of δ which incorporates the Lyapunov functional V as a weight. We record this in the following proposition.

Proposition 3.3 (Weak Harris theorem after Hairer–Mattingly/Eberle)**.** *Assume* (A4) *and let $P := P_{s_t}$ be the skeleton kernel. Fix the projection Π_m and the set C_R from* (A4)(ii)*, and define the truncated projected semimetric δ as above. Then there exist:*

- *a bounded distance-like function $d\colon B \times B \to [0, \infty)$, and*
- *a constant $\kappa \in (0, 1)$,*

depending only on the structural data in (A4)*, such that:*

(i) For all probability measures μ, ν on B with $\mu(V), \nu(V) < \infty$, the Kantorovich functional

$$W_d(\mu, \nu) := \inf_{\pi \in \Gamma(\mu,\nu)} \int_{B \times B} d(x, y)\, \pi(dx, dy)$$

satisfies

$$W_d(\mu P, \nu P) \;\le\; \kappa\, W_d(\mu, \nu). \tag{3.5}$$

(ii) The function d can be chosen so that there exist constants $C_0, \eta > 0$ with

$$d(x, y) \;\le\; C_0\big(\delta(x, y) + \eta\big(V(x) + V(y)\big)\big), \qquad x, y \in B. \tag{3.6}$$

Moreover, the same d and κ work uniformly for any family of Markov kernels whose Lyapunov and minorisation data in (A4) *coincide.*

Remark 3.4. The existence of d and the contraction property (3.5) are standard consequences of the weak Harris theory; see, for instance, the arguments in [9, 10, 6] for general Markov chains on Polish spaces. The precise construction of d uses a concave transformation of the base semimetric δ together with a Lyapunov weight built from V. The upper bound (3.6) reflects the fact that d does not grow faster than a fixed multiple of the projected distance plus a linear function of $V(x)+V(y)$. We will not need the explicit formula for d, only the properties stated above.

We now fix, once and for all, one such distance-like function d as in Proposition 3.3.

Definition 3.5 (Adapted distance and $W_1^{(m)}$)**.** Let d be a bounded distance-like function on B satisfying (3.5) and (3.6) for the skeleton kernel $P := P_{s_t}$, whose existence is guaranteed by Proposition 3.3. For $\mu, \nu \in \mathcal{P}(B)$ with finite V-moment we write

$$W_1^{(m)}(\mu, \nu) := \inf_{\pi \in \Gamma(\mu,\nu)} \int_{B \times B} d(x, y)\, \pi(dx, dy), \tag{3.7}$$

and refer to $W_1^{(m)}$ as the adapted Kantorovich distance associated with the projection Π_m and the Lyapunov function V.

Remark 3.6. The notation $W_1^{(m)}$ emphasises the dependence of the distance on the chosen projection Π_m. In applications one usually fixes m once and for all (e.g. a finite set of low modes of the boundary field) and suppresses this dependence. On sets of probability measures with uniformly bounded V-moment and tightness controlled by the level sets of V, convergence in $W_1^{(m)}$ implies weak convergence. We will not need a more precise characterisation of the topology induced by $W_1^{(m)}$ here.

3.3 Proof of Theorem 2.3

We can now give a short proof of Theorem 2.3.

Proof of Theorem 2.3. Item (i) is exactly the contraction estimate (3.5) of Proposition 3.3, written in terms of $W_1^{(m)}$ via Definition 3.5.

For items (ii) and (iii), we appeal directly to the general weak Harris theory of Hairer–Mattingly and Eberle as formulated, for example, in [9, 10, 6]. Under Assumption (A4), Proposition 3.3 provides a Lyapunov function V, a small set in the sense of (A4)(ii), and a distance-like function d with respect to which P is a strict contraction as in (3.5). The abstract weak Harris theorems in the cited works then imply:

- the existence of at least one invariant probability measure μ with $\mu(V) < \infty$;
- uniqueness of such a measure;
- and the quantitative convergence estimate

$$W_d(\nu P^n, \mu) \le C\,\kappa^n(1 + \nu(V)), \qquad n \in \mathbb{N},$$

for all initial laws ν with finite V-moment, for some $C < \infty$ and $\kappa \in (0,1)$ depending only on the Lyapunov and minorisation data.

Since $W_1^{(m)}$ is defined in terms of d, this is exactly item (iii). Item (ii) is the uniqueness statement. The uniformity in families with common structural data follows from the corresponding uniformity statements in the weak Harris results, which depend only on the constants in (A4). $\square$

4 Structural Hypotheses and Slab Boundary Examples

In this section we introduce a concrete set of structural hypotheses (H1)–(H5) under which the abstract assumptions (A1)–(A4) can be verified for a boundary Langevin dynamics, and we show that these hypotheses imply the Lyapunov drift and projected minorisation conditions for a discrete-time skeleton. We then indicate how (H1)–(H5) are realised for slab boundary laws arising in a constructive approach to four-dimensional SU(N) Yang–Mills theory, and we give a scalar ϕ^4 example for the boundary representation.

4.1 Structural hypotheses for boundary laws

Let $t > 0$ be fixed. We consider, for each choice of external parameters (which one may think of as regulators), an abstract Wiener space (B, H, μ^0), a potential $U \colon B \to \mathbb{R}$, and the associated interacting measure and Langevin dynamics as in Section 2. We assume that the following structural properties hold uniformly in the parameters.

(H1) Ellipticity and quadratic control. The Gaussian covariance C of μ^0 satisfies the ellipticity bounds (2.1) with constants $\kappa_0(t), \kappa_1(t)$ depending only on t. Moreover, the quadratic part of Φ dominates the Cameron–Martin norm uniformly:

$$\frac{1}{2}\langle b, C^{-1}b\rangle_H \;\ge\; \frac{\kappa_0(t)}{2}\,\|b\|_H^2.$$

(H2) Local C^2-regularity and Lipschitz bounds. For each $R > 0$ there exists a constant $C_2(t,R)$ such that U is twice Fréchet differentiable along H on $B_R := \{\|b\|_H \le R\}$ and

$$\sup_{b \in B_R} \|D_H^2 U(b)\|_{\mathrm{op}} \;\le\; C_2(t,R), \tag{4.1}$$

and in particular

$$\|\nabla_H U(b) - \nabla_H U(\tilde b)\|_H \le C_2(t,R)\,\|b - \tilde b\|_H, \qquad b, \tilde b \in B_R.$$

(H3) One-sided growth for the interacting force. There exist nonnegative constants $K_1(t)$, $K_0(t)$ such that

$$\big|\langle b, \nabla_H U(b)\rangle_H\big| \;\le\; K_1(t)\,\|b\|_H + K_0(t), \qquad b \in H. \tag{4.2}$$

Remark 4.1. As in Assumption (A3), condition (H3) plays no direct role in the abstract weak Harris theorem of Section 3. It is included because it arises naturally in concrete SDE models and is convenient when deriving Lyapunov generator bounds such as (4.4) in specific boundary theories.

(H4) Coercive Lyapunov functional. There exists a measurable functional $V_t\colon B \to [1,\infty)$ and constants $\lambda_t > 0$, $C_t \geq 0$, depending only on t, such that:

(i) V_t controls the Cameron–Martin norm polynomially:

$$\alpha_1(t)(1+\|b\|_H^2) \;\leq\; V_t(b) \;\leq\; \alpha_2(t)(1+\|b\|_H^4), \qquad b \in B, \tag{4.3}$$

for some $\alpha_1(t), \alpha_2(t) > 0$;

(ii) the infinitesimal generator $\mathcal{L}$ of the boundary Langevin dynamics satisfies

$$\mathcal{L}V_t(b) \;\leq\; -\lambda_t\, V_t(b) + C_t, \qquad b \in B. \tag{4.4}$$

(H5) Local smoothing and nondegenerate noise on low modes. There exist an integer $m \geq 1$, a bounded linear map $\Pi_m\colon B \to H$ with finite-dimensional range whose restriction to H is an orthogonal projection, a radius $R_0(t) \geq 1$ and a time step $s_t > 0$ such that:

(a) the projected SDE for $\Pi_m B_s$ has a smooth transition density on $\Pi_m H$ for all times $s > 0$;

(b) there exists a probability measure ν_t on $\Pi_m H$ and a constant $\delta_t \in (0,1]$, depending only on t, such that for all initial data $b \in B$ with $V_t(b) \leq R_0(t)$ and all Borel sets $A \subset \Pi_m H$,

$$\mathbb{P}(\Pi_m B_{s_t} \in A \,|\, B_0 = b) \;\geq\; \delta_t\, \nu_t(A).$$

Remark 4.2. Hypothesis (H5)(b) is precisely a small-set Doeblin minorisation for the projected boundary process. In the abstract development we treat this as a structural assumption. In finite-dimensional SDE models it can often be justified by heat-kernel and density estimates for the projected diffusion (see, for instance, Aronson [1] and Stroock–Varadhan [15]), together with regularity of the drift and nondegeneracy of the noise in the low modes. In an infinite-dimensional setting, one typically reduces to a finite-dimensional projected system satisfying a Hörmander-type bracket condition and then uses Malliavin calculus and hypoellipticity to obtain a smooth, strictly positive density on compact sets. Combined with the Lyapunov control provided by (H4), this yields the quantitative small-set estimate in (H5)(b). In the boundary-field setting relevant for constructive QFT, this strategy needs to be adapted to the specific covariance structure and drift, and we refer to model-specific works (e.g. [4]) for those arguments.

The hypotheses (H1)–(H5) abstract the output of a variety of constructive and stochastic analyses in concrete models. In particular, (H4) encodes the existence of a coercive Lyapunov functional and (H5) captures a quantitative small-set condition for a finite number of low modes.

4.2 Lyapunov drift for the skeleton chain

We first derive the discrete-time Lyapunov drift condition (A4)(i) starting from the generator inequality (4.4).

Proposition 4.3 (Lyapunov drift for the skeleton chain)**.** *Fix $t > 0$ and assume* (H4)*. Let $s_t > 0$ be as in* (H5) *and set $P := P_{s_t}$ and $X_n := B_{ns_t}$. Then there exist constants $a_t \in (0,1)$ and $c_t \geq 0$, depending only on t, such that for all initial conditions $b \in B$,*

$$PV_t(b) := \mathbb{E}\big[V_t(X_1) \mid X_0 = b\big] \;\leq\; a_t\, V_t(b) + c_t. \tag{4.5}$$

In particular, the Lyapunov drift Assumption (A4)(i) *holds with $V := V_t$, $a := a_t$ and $c := c_t$.*

Proof. Let $\mathcal{L}$ be the infinitesimal generator of the boundary Langevin semigroup $(P_s)_{s\geq 0}$. For functions f with suitable growth controlled by V_t (in particular $f = V_t$), Dynkin's formula yields

$$P_sV_t(b) - V_t(b) = \mathbb{E}_b\big[V_t(B_s) - V_t(B_0)\big] = \mathbb{E}_b\Big[\int_0^s \mathcal{L}V_t(B_r)\, dr\Big], \tag{4.6}$$

where $\mathbb{E}_b$ denotes expectation for the process started at $B_0 = b$.

Using (4.4) we obtain

$$\mathcal{L}V_t(B_r) \leq -\lambda_t V_t(B_r) + C_t$$

for all $r \geq 0$. Substituting into (4.6) we get

$$P_sV_t(b) - V_t(b) \leq -\lambda_t \int_0^s \mathbb{E}_b[V_t(B_r)]\, dr + C_t s. \tag{4.7}$$

Define

$$m_b(s) := \mathbb{E}_b[V_t(B_s)], \qquad s \geq 0.$$

Then (4.7) reads

$$m_b(s) - V_t(b) \leq -\lambda_t \int_0^s m_b(r)\, dr + C_t s.$$

Differentiating in s yields

$$m_b'(s) \leq -\lambda_t\, m_b(s) + C_t, \qquad s > 0, \tag{4.8}$$

with initial condition $m_b(0) = V_t(b)$.

Solving the corresponding equality gives

$$m_b(s) = e^{-\lambda_t s} V_t(b) + \frac{C_t}{\lambda_t}(1 - e^{-\lambda_t s}),$$

and the comparison principle for linear scalar differential inequalities implies

$$m_b(s) \leq e^{-\lambda_t s} V_t(b) + \frac{C_t}{\lambda_t}(1 - e^{-\lambda_t s}), \qquad s \geq 0. \tag{4.9}$$

For $s = s_t$ we obtain

$$P_{s_t}V_t(b) = m_b(s_t) \leq e^{-\lambda_t s_t} V_t(b) + \frac{C_t}{\lambda_t}(1 - e^{-\lambda_t s_t}).$$

Define

$$a_t := e^{-\lambda_t s_t} \in (0,1), \qquad c_t := \frac{C_t}{\lambda_t}.$$

This yields (4.5) and shows that (A4)(i) holds with the stated choices. $\square$

4.3 Verification of Assumption (A4)

We summarise the preceding analysis in a single proposition.

Proposition 4.4 (Verification of (A4))**.** *Fix $t > 0$ and consider the skeleton chain $X_n := B_{ns_t}$ with transition kernel $P := P_{s_t}$ on B. Assume* (H1)–(H5) *hold. Then Assumption* (A4) *is satisfied with the following choices:*

- *Lyapunov function $V := V_t$ and constants $a = a_t \in (0, 1)$, $c = c_t \geq 0$ from Proposition 4.3;*
- *small set $C_R := \{b \in B : V_t(b) \leq R_0(t)\}$ with radius $R := R_0(t)$;*
- *bounded linear map $\Pi_m : B \to H$ and probability measure ν_t on $\Pi_m H \cong \mathbb{R}^m$ as in (H5);*
- *minorisation constant $\delta_0 := \delta_t \in (0, 1]$ from (H5)(b);*
- *coercivity parameters α_1, α_2, p inherited from* (H4)(i)*.*

Proof. The Lyapunov drift inequality (A4)(i) follows from Proposition 4.3, with $V := V_t$, $a := a_t$ and $c := c_t$. The projected minorisation (A4)(ii) is exactly (H5)(b), with $R_0(t)$, Π_m, ν_t and δ_t as specified there. The coercivity (A4)(iii) follows directly from (H4)(i), with $p = 4$. □

Combining Proposition 4.4 with Theorem 2.3, we obtain:

Corollary 4.5 (Harris mixing for boundary laws under (H1)–(H5))**.** *Fix $t > 0$ and assume* (H1)–(H5)*. Then there exist constants $\kappa_{s_t} \in (0, 1)$ and $C_{s_t} < \infty$, depending only on t, such that for the time–s_t skeleton $X_n := B_{ns_t}$ of the boundary Langevin dynamics there is a unique invariant probability measure μ with $\mu(V_t) < \infty$, and for any initial law $\nu \in \mathcal{P}(B)$ with $\nu(V_t) < \infty$,*

$$W_1^{(m)}(\nu P^n, \mu) \;\leq\; C_{s_t}\, \kappa_{s_t}^n \,(1 + \nu(V_t)), \qquad n \in \mathbb{N},$$

where $P = P_{s_t}$ is the skeleton kernel and $W_1^{(m)}$ is the adapted Kantorovich distance defined in Definition 3.5.

4.4 Example: slab boundary laws for Yang–Mills theory

The purpose of this subsection is illustrative: detailed proofs for the Yang–Mills case are provided in the companion work [4]. We outline the key structures to show how the abstract hypotheses (H1)–(H5) arise in a gauge-theoretic setting.

For each slab thickness $t > 0$ and regulators (L, Λ, M) describing the spatial volume, ultraviolet cutoff, and mass or gauge-fixing parameters, the boundary field on $\{0\} \times \mathbb{T}_L^3$ is realised as a random variable $A_{t,L,\Lambda}$ in an abstract Wiener space $(B_{t,\Lambda}, H_{t,\Lambda}, \mu^0_{t,\Lambda})$. Here $H_{t,\Lambda}$ is a Sobolev-type Cameron–Martin space of transverse $\mathfrak{su}(N)$-valued one-forms on the spatial torus $\mathbb{T}_L^3$, and $B_{t,\Lambda}$ is a suitable Banach completion (e.g. based on a Besov or Hölder norm). The Gaussian reference measure $\mu^0_{t,\Lambda}$ is a massive, gauge-fixed Gaussian field with covariance operator $C^0_{t,\Lambda}$ obtained from the linearised Yang–Mills action with ultraviolet cutoff Λ.

The interacting boundary law $\mu_{t,L,\Lambda}$ has density

$$\frac{d\mu_{t,L,\Lambda}}{d\mu^0_{t,\Lambda}}(A) \;=\; Z^{-1}_{t,L,\Lambda}\, \exp(-U_{t,L,\Lambda}(A)),$$

where the potential $U_{t,L,\Lambda}$ collects the quartic and higher-order gauge interactions, the effects of integrating out bulk degrees of freedom in the slab of thickness t, and regulator-dependent counterterms.

A Lyapunov functional V_t is constructed in [4] as a polynomially coercive, gauge-invariant energy of the schematic form

$$V_t(A) \;\simeq\; 1 + \|A\|_{H^1}^4 + \int_{\mathbb{T}_L^3} |F_A(x)|^2\, dx,$$

where F_A denotes the curvature of A. Finite-range covariance decomposition and mixed quartic–gradient coercivity estimates for the interacting action imply polynomial bounds of the form

$$\alpha_1(t)\big(1 + \|A\|_{H^1}^2\big) \;\leq V_t(A) \;\leq\; \alpha_2(t)\big(1 + \|A\|_{H^1}^4\big),$$

with constants independent of (L, Λ, M) for fixed $t > 0$. The boundary Langevin generator associated with $U_{t,L,\Lambda}$ satisfies a drift inequality

$$\mathcal{L}V_t(A) \;\leq\; -\lambda_t V_t(A) + C_t,$$

with $\lambda_t > 0$ and $C_t \geq 0$ uniform in (L, Λ, M); see [4] for details. This verifies (H4).

For (H5), one projects onto finitely many spatial Fourier modes of the boundary gauge potential,

$$\Pi_m A(x) := \sum_{|k|\leq m} \hat{A}(k) e^{ik\cdot x},$$

and uses the nondegeneracy of the Gaussian reference covariance $C^0_{t,\Lambda}$ in these modes together with the local Lipschitz bounds on the interacting drift provided by (H2). The projected SDE for $\Pi_m A_s$ is a finite-dimensional nondegenerate diffusion with smooth coefficients, and its transition density at time s_t admits lower and upper Gaussian-type bounds that are uniform for A restricted to a fixed V_t-sublevel set. Combining these density bounds with the Lipschitz estimate on the small set yields a Doeblin-type minorisation of the form (H5)(b), with constants $\delta_t > 0$ and a reference measure ν_t independent of (L, Λ, M); see [4] for details.

In this way the slab boundary laws for Yang–Mills provide a concrete family of examples where the hypotheses of Corollary 4.5 are satisfied uniformly in the ultraviolet, infrared, and volume regulators, for each fixed slab thickness $t > 0$.

4.5 Remark on scalar field models and a worked example

A similar set of structural hypotheses can be verified for massive scalar ϕ^4 theory in finite volume. In that setting, the Gaussian reference measure is the free massive field with covariance $(-\Delta + m^2)^{-1}$ on a spatial torus or box, and the potential U is the usual ϕ^4 interaction plus counterterms. Coercive Lyapunov functionals of the form $V_t(b) \sim 1 + \|b\|_{H^1}^4$ arise naturally from the massive Gaussian tail behaviour and the quartic interaction, and finite-range decomposition combined with cluster and polymer expansions provides the local regularity and small-set properties required in (H2)–(H5). Classical references include the monographs [7, 14], Brydges' lectures [3], and the cluster expansion analyses of Feldman, Osterwalder and Hurd in the $P(\phi)_2$ setting, which are emblematic of the techniques used to control effective interactions and quasi-locality in constructive field theory.

To make Assumption 5.1 more tangible, we sketch one explicit example of a boundary representation and Lipschitz control for a simple bounded scalar observable.

Proposition 4.6 (Example of Assumption 5.1 in massive scalar ϕ^4)**.** *Consider massive scalar ϕ^4 theory on a finite Euclidean volume with periodic boundary conditions, and fix a slab of thickness*

$t > 0$. Let $\mathcal{O}(\phi) = F(\phi(g))$ be a bounded cylinder observable supported in a single slab, where g is a smooth compactly supported test function and $F \in C_b^1(\mathbb{R})$ is bounded with bounded derivative. Then there exists a boundary functional $F_{\mathcal{O}} \colon B \to \mathbb{R}$ such that

$$\mathcal{O}(\phi) = F_{\mathcal{O}}(B_0(\phi))$$

almost surely, and constants $L_{\mathcal{O}}, C_{\mathcal{O}} \geq 0$ such that the Lipschitz and boundedness conditions of Assumption 5.1 hold for $F_{\mathcal{O}}$.

Proof sketch. We outline the standard constructive argument, adapted to the present setting. The slab measure with fixed boundary field b at time 0 can be constructed by a finite-range covariance decomposition and a convergent cluster or polymer expansion around a Gaussian reference field with mean determined by b; see, for instance, [7, Chs. XI–XII], as well as Brydges [3] and related cluster expansion works in the spirit of Feldman–Osterwalder–Hurd. Integrating out the bulk degrees of freedom in the slab yields an effective boundary action and a representation of $\mathcal{O}(\phi)$ as a functional $F_{\mathcal{O}}(b)$ of the boundary configuration b.

The multiscale decomposition and finite-range property imply that the dependence of $F_{\mathcal{O}}(b)$ on b is quasi-local in space: variations of b far from the spatial support of g have exponentially small influence on $F_{\mathcal{O}}(b)$, and the influence of local variations in b can be expressed in terms of a convergent series of polymer activities. Differentiating the polymer expansion termwise with respect to b shows that the Fréchet derivative $DF_{\mathcal{O}}(b)$ exists and is bounded in a way that is uniform on V_t-sublevel sets, so the Lipschitz bound

$$|F_{\mathcal{O}}(b) - F_{\mathcal{O}}(\tilde{b})| \leq L_{\mathcal{O}}\, d(b, \tilde{b})$$

holds for a suitable constant $L_{\mathcal{O}}$. Since $\mathcal{O}$ is bounded, we also have a global bound

$$|F_{\mathcal{O}}(b)| \leq C_{\mathcal{O}} := \|\mathcal{O}\|_{\infty},$$

which is precisely the boundedness condition in Assumption 5.1. We refer to [7, 14, 3] for detailed proofs of the cluster and quasi-locality estimates on which this sketch is based. $\square$

Remark 4.7. In the scalar setting one expects Assumption 5.1 to hold for a much wider class of observables than those covered by Proposition 4.6, including polynomials in smeared fields and their spatial derivatives, as well as products of such observables localised in a slab. Cylinder observables of the form treated above already generate a dense subalgebra of local fields in the OS Hilbert space, so for the purposes of OS4 it suffices to control this class. Extending the boundary representation and Lipschitz control to more general observables is a matter of refining the quasi-locality and cluster expansion estimates, and follows the same structural pattern. In gauge theories such as Yang–Mills, one expects an analogous boundary representation for gauge-invariant observables built from Wilson loops and smeared local polynomials in the curvature, but the verification is technically more involved due to gauge symmetry and topology; these issues are addressed in the model-specific analysis in [4].

This example illustrates that Assumption 5.1 is not merely formal: in standard constructive scalar models one can verify it by combining quasi-locality and cluster expansions with a Lyapunov functional controlling the boundary field.

5 OS4 via Slab Concatenation and OS Reconstruction

In this section we derive the Osterwalder–Schrader exponential clustering axiom (OS4) for Euclidean field measures that admit a slab Markov decomposition and satisfy the structural hypotheses of the previous sections. The strategy is:

(i) express time-separated local observables as functionals of boundary data on a discrete time grid $\{0, t, 2t, \dots\}$ using a boundary representation;

(ii) apply the Harris mixing result of Corollary 4.5 to obtain exponential decay of covariances for these boundary functionals;

(iii) translate the resulting mixing estimate into an OS4 bound on Schwinger function covariances via the standard OS reconstruction machinery.

Throughout this section we fix a slab thickness $t > 0$. We recall that the Harris analysis is performed for the time–s_t skeleton with kernel $P = P_{s_t}$, and we assume from now on that there exists an integer $m_t \geq 1$ such that

$$t = m_t s_t. \tag{5.1}$$

The Markov kernel governing the boundary chain sampled at integer multiples of t will then be

$$\mathsf{T}_t := P^{m_t}. \tag{5.2}$$

5.1 Slab tower, transfer kernel and OS measure

We assume that for each choice of external parameters there is a consistent family of Euclidean field measures $\{\mathbb{P}^{(n)}\}_{n\in\mathbb{N}}$ on fields defined on finite time intervals $[0, nt]$ obtained by concatenating n slabs of thickness t, with the following properties:

- **Reflection positivity and time-translation invariance.** Each $\mathbb{P}^{(n)}$ is reflection positive with respect to time reflection about a hyperplane $\{\tau = kt\}$, and invariant under integer time translations $\tau \mapsto \tau + \ell t$, $\ell \in \mathbb{Z}$.

- **Markov property in slab time.** The joint law of the field on the slab tower $[0, nt]$ is Markov with respect to the discrete time filtration given by the boundaries at times $\{0, t, \dots, nt\}$: conditionally on these boundaries, the interior fields in distinct slabs are independent.

- **Boundary marginal and transfer kernel.** The law of the boundary configuration at time kt under $\mathbb{P}^{(n)}$ coincides with the stationary law μ of the boundary chain of Corollary 4.5, and the conditional law of the boundary at time $(k+1)t$ given that at time kt is described by the transfer kernel T_t defined in (5.2).

Passing to a projective limit in n yields an OS measure $\mathbb{P}$ on fields on the whole time axis with the usual OS0–OS3 properties. The OS reconstruction then produces a Hilbert space, a vacuum vector, and a nonnegative self-adjoint Hamiltonian generating time translations; we recall this briefly next.

5.2 OS reconstruction

Let $\mathcal{F}_+$ be the algebra of bounded, local observables supported in the half-space $\{\tau \geq 0\}$. For $F, G \in \mathcal{F}_+$, define the pre-inner product

$$\langle F, G\rangle_{\mathrm{OS}} := \mathbb{E}_{\mathbb{P}}[\,\Theta F \cdot G\,],$$

where Θ is time reflection about $\{\tau = 0\}$. Quotienting by the null space $\{F : \langle F, F\rangle_{\mathrm{OS}} = 0\}$ and completing yields the OS Hilbert space H_{OS}; we denote the equivalence class of $F \in \mathcal{F}_+$ by $[F]$, and the vacuum vector by $\Omega := [\mathbb{1}]$.

Time translations by integer multiples of t define operators $\tau_{nt} \colon \mathcal{F}_+ \to \mathcal{F}_+$, $n \in \mathbb{N}$, given by $(\tau_{nt}F)(\phi) := F(\phi(\cdot + nt))$. By the OS axioms, the map τ_{nt} descends to a contraction semigroup $(T_{nt})_{n\in\mathbb{N}}$ on H_{OS} via

$$T_{nt}[F] := [\tau_{nt}F].$$

Standard OS reconstruction (see [12, 13, 7]) implies that there exists a nonnegative self-adjoint operator H on H_{OS} and a strongly continuous contraction semigroup $(e^{-sH})_{s\geq 0}$ such that:

(i) $T_{nt} = e^{-ntH}$ for all $n \in \mathbb{N}$;

(ii) Ω is a cyclic, invariant vector, i.e. $e^{-sH}\Omega = \Omega$ for all $s \geq 0$;

(iii) time-ordered Schwinger functions can be written as vacuum expectations of products of translated operators.

5.3 Boundary representation of slab observables

We now formalise the assumption that time-separated local observables can be represented as boundary functionals at discrete times, with Lipschitz and boundedness properties compatible with the Harris structure.

Assumption 5.1 (Boundary representation and Lipschitz control)**.** Fix $t > 0$. For each bounded local observable $\mathcal{O}$ supported in a single slab $[\ell t, (\ell + 1)t] \times \Lambda_{\mathrm{sp}} \subset \mathbb{R} \times \mathbb{R}^3$ with $\ell \in \mathbb{Z}$ and spatial support Λ_{sp}, there exists a measurable function

$$F_{\mathcal{O}} \colon B \to \mathbb{R}$$

and a constant $L_{\mathcal{O}} \in (0, \infty)$ such that:

(i) *Boundary representation.* By time-translation invariance we may shift the support of $\mathcal{O}$ into the slab $[0, t] \times \Lambda_{\mathrm{sp}}$. For every choice of external parameters we then have

$$\mathcal{O}(\phi) = F_{\mathcal{O}}(B_0(\phi)) \tag{5.3}$$

almost surely under the corresponding Euclidean field measure $\mathbb{P}$, where $B_0(\phi)$ denotes the boundary configuration at time 0 obtained from the field ϕ by restriction to $\{0\} \times \mathbb{R}^3$.

(ii) *Lipschitz and boundedness.* The map $F_{\mathcal{O}}$ is Lipschitz with respect to the adapted distance-like function d of Definition 3.5, i.e.

$$|F_{\mathcal{O}}(b) - F_{\mathcal{O}}(\tilde{b})| \;\leq\; L_{\mathcal{O}}\, d(b, \tilde{b}), \qquad b, \tilde{b} \in B, \tag{5.4}$$

and there exists a constant $C_{\mathcal{O}} \geq 0$ such that

$$|F_{\mathcal{O}}(b)| \;\leq\; C_{\mathcal{O}}, \qquad b \in B. \tag{5.5}$$

Moreover, for observables whose spatial support is contained in a fixed compact set Λ_{sp}, the constants $L_{\mathcal{O}}, C_{\mathcal{O}}$ can be chosen uniformly in any family of models under consideration.

5.4 From boundary Harris mixing to covariance decay

Let $(X_n)_{n\geq 0}$ denote the time–s_t skeleton of the boundary Langevin dynamics with invariant law μ, as in Corollary 4.5. Using the integer $m_t \geq 1$ from (5.1), we define a new chain sampled at the slab times

$$\widehat{X}_k := B_{kt} = X_{km_t}, \qquad k \in \mathbb{N}.$$

The Markov kernel governing $(\widehat{X}_k)_{k\geq 0}$ is the iterate

$$\mathsf{T}_t := P^{m_t}$$

defined in (5.2). Corollary 4.5 gives, for some constants $\kappa_{s_t} \in (0,1)$ and $C_{s_t} < \infty$ depending only on t,

$$W_1^{(m)}(\nu P^n, \mu) \leq C_{s_t}\, \kappa_{s_t}^n\, (1 + \nu(V_t)), \qquad n \in \mathbb{N},$$

for all probability measures ν with $\nu(V_t) < \infty$. For convenience we now set

$$\kappa_t := \kappa_{s_t}^{m_t} \in (0,1), \qquad C_t := C_{s_t},$$

so that, for the t-step transfer kernel T_t,

$$W_1^{(m)}(\nu \mathsf{T}_t^n, \mu) = W_1^{(m)}(\nu P^{nm_t}, \mu) \leq C_t\, \kappa_t^n\, (1 + \nu(V_t)), \qquad n \in \mathbb{N}. \tag{5.6}$$

We now translate this into a quantitative covariance decay for boundary functionals.

Lemma 5.2 (Covariance decay for boundary functionals)**.** *Let Assumptions* (A1)–(A4) *and 5.1 hold, and let $F_1, F_2 \colon B \to \mathbb{R}$ be functions satisfying the Lipschitz and boundedness conditions* (5.4)–(5.5)*. Then there exist constants $C < \infty$ and $\gamma \in (0,1)$, depending only on the Harris data and on the Lipschitz and sup-norm bounds of F_1, F_2, such that for all $n \in \mathbb{N}$,*

$$\left|\mathrm{Cov}_\mu(F_1(\widehat{X}_0), F_2(\widehat{X}_n))\right| \;\leq\; C\,\gamma^n. \tag{5.7}$$

Proof. Let $W_1^{(m)}$ denote the adapted Kantorovich distance associated with d, and let $\kappa_t \in (0,1)$ and $C_t < \infty$ be as in (5.6) for the t-step kernel T_t. For $b \in B$, write

$$\mathcal{L}_b^{(n)} := \mathcal{L}(\widehat{X}_n \,|\, \widehat{X}_0 = b) = \delta_b \mathsf{T}_t^n$$

for the conditional law of $\widehat{X}_n$ given $\widehat{X}_0 = b$.

By the Lipschitz property (5.4) we have, for each $b \in B$,

$$\left|\mathbb{E}[F_2(\widehat{X}_n) \mid \widehat{X}_0 = b] - \mathbb{E}_\mu[F_2]\right| \leq \mathrm{Lip}_d(F_2)\, W_1^{(m)}(\mathcal{L}_b^{(n)}, \mu).$$

Using (5.6) with $\nu = \delta_b$ gives

$$W_1^{(m)}(\mathcal{L}_b^{(n)}, \mu) = W_1^{(m)}(\delta_b \mathsf{T}_t^n, \mu) \leq C_t\, \kappa_t^n\, (1 + V_t(b)),$$

and hence

$$\left|\mathbb{E}[F_2(\widehat{X}_n) \mid \widehat{X}_0 = b] - \mathbb{E}_\mu[F_2]\right| \leq C_t\, \mathrm{Lip}_d(F_2)\, \kappa_t^n\, (1 + V_t(b)), \qquad b \in B. \tag{5.8}$$

Now write the covariance as

$$\mathrm{Cov}_\mu(F_1(\widehat{X}_0), F_2(\widehat{X}_n)) = \int_B F_1(b) \left(\mathbb{E}[F_2(\widehat{X}_n) \mid \widehat{X}_0 = b] - \mathbb{E}_\mu[F_2]\right) \mu(db).$$

Using (5.8) and the bound $|F_1| \le \|F_1\|_\infty$ we obtain

$$\big|\mathrm{Cov}_\mu(F_1(\widehat{X}_0), F_2(\widehat{X}_n))\big| \le \|F_1\|_\infty \, C_t \, \mathrm{Lip}_d(F_2)\, \kappa_t^n \int_B (1 + V_t(b))\, \mu(db).$$

Since μ is invariant for the skeleton chain and V_t satisfies the Lyapunov drift inequality $PV_t \le a_t V_t + c_t$, integrating against μ yields

$$\mu(V_t) = \int V_t \, d\mu = \int PV_t \, d\mu \le a_t \mu(V_t) + c_t,$$

so $(1 - a_t)\mu(V_t) \le c_t$ and hence $\mu(V_t) < \infty$. Thus the integral $\int_B (1 + V_t(b))\, \mu(db)$ is finite, and we can absorb it into the prefactor C.

Defining

$$\gamma := \kappa_t \in (0,1), \qquad C := \|F_1\|_\infty \, C_t \, \mathrm{Lip}_d(F_2) \int_B (1 + V_t(b))\, \mu(db),$$

we obtain (5.7). This proves the lemma. □

5.5 OS4 for slab-supported local observables

We now combine the boundary covariance decay of Lemma 5.2 with the boundary representation of Assumption 5.1 and the OS reconstruction to obtain OS4 for time-separated local observables.

Let $\mathcal{O}_1, \mathcal{O}_2$ be bounded local observables supported in slabs of thickness t centred at times $\tau_1, \tau_2 \in \mathbb{R}$ with $\tau_2 - \tau_1 = nt$ for some $n \in \mathbb{N}$. Without loss of generality, we may assume $\tau_1 \in [0, t]$, by translational invariance, and then $\tau_2 \in [nt, (n+1)t]$.

By Assumption 5.1, there exist functions $F_{\mathcal{O}_1}, F_{\mathcal{O}_2} \colon B \to \mathbb{R}$ such that

$$\mathcal{O}_1(\phi) = F_{\mathcal{O}_1}(B_0(\phi)), \qquad \mathcal{O}_2(\phi) = F_{\mathcal{O}_2}(B_{nt}(\phi))$$

almost surely under $\mathbb{P}$.

Using stationarity and the Markov property one checks that

$$\mathrm{Cov}_{\mathbb{P}}(\mathcal{O}_1, \Theta_{nt}\mathcal{O}_2) = \mathrm{Cov}_\mu\Big(F_{\mathcal{O}_1}(\widehat{X}_0), F_{\mathcal{O}_2}(\widehat{X}_n)\Big),$$

where Θ_T denotes time translation by T and $(\widehat{X}_k)$ is the boundary chain sampled every time t. Applying Lemma 5.2 with $F_1 = F_{\mathcal{O}_1}$, $F_2 = F_{\mathcal{O}_2}$ yields

$$\big|\mathrm{Cov}_{\mathbb{P}}(\mathcal{O}_1, \Theta_{nt}\mathcal{O}_2)\big| \le C_{\mathcal{O}_1,\mathcal{O}_2}\, \kappa_t^n,$$

where $C_{\mathcal{O}_1,\mathcal{O}_2}$ depends only on the Lipschitz and sup-norm bounds of $F_{\mathcal{O}_1}, F_{\mathcal{O}_2}$ and on the Harris data. Writing $T = nt$ and setting

$$\rho(t) := -\frac{1}{t}\log \kappa_t > 0,$$

we arrive at the OS4-type bound

$$\big|\mathrm{Cov}_{\mathbb{P}}(\mathcal{O}_1, \Theta_T\mathcal{O}_2)\big| \;\le\; C_{\mathcal{O}_1,\mathcal{O}_2}\, e^{-\rho(t)T}, \qquad T \in t\mathbb{N}. \tag{5.9}$$

Proposition 5.3 (OS4 for slab-supported observables)**.** *Fix $t > 0$ and assume* (H1)–(H5) *and Assumption 5.1. Then there exists $\rho(t) > 0$ such that for every pair of bounded local observables $\mathcal{O}_1, \mathcal{O}_2$ supported in slabs of thickness t separated by a time distance $T = nt$, $n \in \mathbb{N}$, the covariance bound* (5.9) *holds.*

Proof. The argument above gives exactly (5.9), with $\rho(t) > 0$ defined in terms of $\kappa_t \in (0,1)$ and with $C_{\mathcal{O}_1,\mathcal{O}_2}$ depending only on the Lipschitz and sup-norm bounds of the boundary representatives and on the Harris data. □

5.6 Extension from discrete to continuous times

The covariance estimate in Proposition 5.3 is proved for $T \in t\mathbb{N}$, corresponding to integer iterates of the transfer operator $T_t = e^{-tH}$. We now explain how to extend this to all $T > 0$ while preserving the exponential decay rate $\rho(t)$, at the expense of a harmless adjustment of the prefactor.

Let $\mathcal{O}_1, \mathcal{O}_2$ be bounded local observables supported in slabs of thickness t and consider their OS representatives A_1, A_2 as operators on H_{OS}. We assume throughout that such observables define bounded operators on H_{OS}, so that $A_i\Omega \in H_{\mathrm{OS}}$ and the norms $\|A_i\Omega\|$ are finite. Then

$$\mathrm{Cov}_{\mathbb{P}}(\mathcal{O}_1, \Theta_T\mathcal{O}_2) = \langle \Omega, A_1 e^{-TH} A_2\Omega\rangle_c,$$

where $\langle\cdot,\cdot\rangle_c$ denotes the connected part of the vacuum expectation. Write $T = nt + \delta$ with $n \in \mathbb{N}$ and $\delta \in [0, t)$. Then

$$A_1 e^{-TH} A_2 = A_1 e^{-\delta H} e^{-ntH} A_2.$$

By strong continuity and contractivity of $(e^{-\delta H})_{\delta \geq 0}$ we have $\|e^{-\delta H}\|_{\mathcal{B}(H_{\mathrm{OS}})} \leq 1$ for all $\delta \geq 0$, so

$$\left|\mathrm{Cov}_{\mathbb{P}}(\mathcal{O}_1, \Theta_T\mathcal{O}_2)\right| = \left|\langle \Omega, A_1 e^{-\delta H} e^{-ntH} A_2\Omega\rangle_c\right| \leq \left|\langle \Omega, A_1 e^{-ntH} A_2\Omega\rangle_c\right| = \left|\mathrm{Cov}_{\mathbb{P}}(\mathcal{O}_1, \Theta_{nt}\mathcal{O}_2)\right|.$$

By Proposition 5.3, the right-hand side is bounded by $C_{\mathcal{O}_1,\mathcal{O}_2}\kappa_t^n$, so

$$\left|\mathrm{Cov}_{\mathbb{P}}(\mathcal{O}_1, \Theta_T\mathcal{O}_2)\right| \leq C_{\mathcal{O}_1,\mathcal{O}_2}\,\kappa_t^n.$$

Since $T \in [nt, (n+1)t)$, we have $n \geq T/t - 1$, and therefore

$$\kappa_t^n \leq \kappa_t^{T/t-1} = \kappa_t^{-1} e^{-\rho(t)T}.$$

Absorbing the factor κ_t^{-1} into the prefactor yields

$$\left|\mathrm{Cov}_{\mathbb{P}}(\mathcal{O}_1, \Theta_T\mathcal{O}_2)\right| \leq C'_{\mathcal{O}_1,\mathcal{O}_2}\, e^{-\rho(t)T}, \qquad T > 0, \tag{5.10}$$

with $C'_{\mathcal{O}_1,\mathcal{O}_2} := C_{\mathcal{O}_1,\mathcal{O}_2}\kappa_t^{-1}$. The decay rate remains $\rho(t)$.

Corollary 5.4 (OS4 in the sense of the OS axioms)**.** *Under the hypotheses of Proposition 5.3, the family of Schwinger functions associated with $\mathbb{P}$ satisfies the OS4 exponential clustering axiom* (5.10) *with some $\rho(t) > 0$.*

Remark 5.5 (On the dependence on t and the mass gap)**.** It is important to emphasise the role of the slab thickness $t > 0$ in the constants. The Harris constant $\kappa_t \in (0, 1)$ obtained above depends on t, and so does the decay rate

$$\rho(t) = -\frac{1}{t}\log \kappa_t.$$

Our framework shows that for each fixed $t > 0$ one can choose the slab decomposition and structural data so that $\rho(t) > 0$, uniformly in the external regulators of a given constructive model. In terms of the reconstructed Hamiltonian H, the bound (5.10) implies that the spectral measure of H in the sector generated by slab-supported observables has a gap of at least $\rho(t)$ above 0, so $\rho(t)$ is a lower bound on the physical mass gap for that choice of t. However, we do not attempt here to optimise over t or to identify the actual mass gap

$$m_{\mathrm{phys}} := \inf\big(\sigma(H) \setminus \{0\}\big),$$

which in general requires additional spectral analysis and, in constructive settings, delicate control of the $t \to 0$ and regulator limits. In applications, one usually combines the present Harris/OS4 engine with a more detailed study of the spectrum of H in order to identify or bound m_{phys}; this lies beyond the scope of this paper.

In concrete constructive models such as the Yang–Mills slab measures of [4], one can often pass to an infinite-volume and continuum limit while preserving reflection positivity and the OS0–OS3 axioms. Since $\rho(t)$ and the prefactors $C_{\mathcal{O}_1,\mathcal{O}_2}$ are expressed purely in terms of the Harris and boundary-representation data, they are stable under such limits as long as these data remain uniform in the regulators. In that setting, the OS4 bound survives in the limiting theory with the same decay rate $\rho(t)$, and the framework presented here can be used as a modular Harris/OS4 engine inside larger constructive programmes.

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